MATH 124 Examination 3
November 25, 2008

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.

1. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 1 km^2/hour. How fast is the radius of the spill increasing when the area of the circle is 10 km^2.

2. Your goal in this problem is to make the cheapest rectangular box with a square base without the top. The volume of the box should be 1 cubic foot. Since you want the cheapest box you should use the least amount of material for five sides. All sides are made of the same material. Determine the dimensions of this box.

3. Find positive real numbers a and b such that the function $f(x) = a\,x\,b^x$ has a local maximum at the point $(2, 1)$

4. Find the tangent lines to the parabola $y = x^2$ that go through the point $(1, -1)$.

5. Use L'Hopital's Rule to find the following limits:

$$\text{(A)} \quad \lim_{x\to 1} \frac{3^x - 3}{2^x - 2}, \qquad\qquad \text{(B)} \quad \lim_{x\to 0}\, (1+x)^{1/x}.$$

$\boxed{1}$

(1) $\dfrac{dA}{dt} = 1$

$A = 10$

$\dfrac{dr}{dt} = ?$

$A = r^2 \pi$

$\dfrac{dA}{dt} = 2r \dfrac{dr}{dt} \pi$

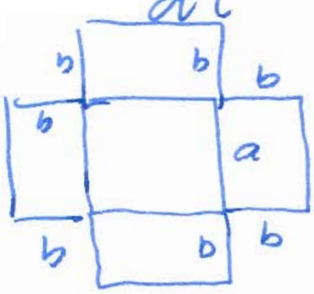
$10 = r^2 \pi \Rightarrow r = \sqrt{\dfrac{10}{\pi}}$

$1 = 2 \cdot \sqrt{\dfrac{10}{\pi}} \dfrac{dr}{dt} (\sqrt{\pi})^2$

$1 = 2 \cdot \sqrt{10\pi} \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{1}{2\sqrt{10\pi}} \approx 0.0631$

(2)

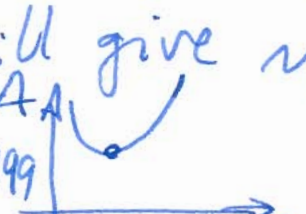


$V = a^2 b = 1, \quad b = \dfrac{1}{a^2}$

$A = a^2 + 4ab$

$A(a) = a^2 + 4\dfrac{1}{a}, \quad A'(a) = 2a - \dfrac{4}{a^2} = \dfrac{2}{a^2}(a^3 - 2)$ neg, 0, pos

1.2599 $a = \sqrt[3]{2} \approx 1.2599$ will give minimum area



$0.6299 \approx b = \dfrac{1}{\sqrt[3]{4}} \approx 0.6299$

$= a/2$

(3) $f(x) = axb^x$

$f'(x) = ab^x + ax(\ln b)b^x$ $\boxed{2}$

$= ab^x(1 + x \ln b)$ $\to$ $-1/2$

We need $1 + x\ln b = 0$ $\boxed{\text{pos, 0, neg}}$

so $x = -\dfrac{1}{\ln b}$

but $x = 2$ so $2 = -\dfrac{1}{\ln b}$

$\ln b = -1/2$

$\boxed{b = \dfrac{1}{\sqrt{e}}}$

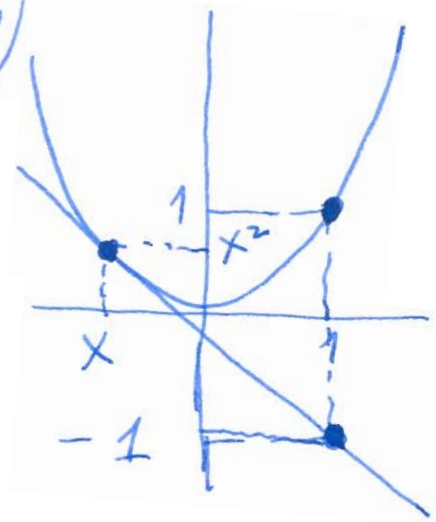
$f(2) = 1$

$a \cdot 2 \cdot b^2 = 1$ $\qquad b = e^{-1/2}$

$a \cdot 2 \cdot \underbrace{(e^{-1/2})^2}_{e^{-1}} = 1$, $\boxed{a = \dfrac{e}{2}}$

$f(x) = \dfrac{e}{2} e^{-x/2} = \dfrac{e}{2}\left(\dfrac{1}{\sqrt{e}}\right)^x$

④



$\boxed{3}$

$$\text{Find } \underbrace{(x, x^2)}_{\text{a point}}$$

such that the line from $(1,-1)$ to (x, x^2) has the slope $2x$. Solve

$$2x = \frac{x^2-(-1)}{x-1} \text{ for } x$$

$$2x^2 - 2x = x^2 + 1$$

$$x^2 - 2x - 1 = 0$$

$$x^2 - 2x + 1 - 2 = 0$$

$$(x-1)^2 = 2 \qquad x - 1 = \pm\sqrt{2}$$

$$\boxed{x = 1 \pm \sqrt{2}}$$

So, the slopes are $2(1+\sqrt{2})$ and $2(1-\sqrt{2})$

lines are

$$\boxed{y = 2(1 \pm \sqrt{2})(x-1) - 1}$$

5

derivative

$$\lim_{x\to 1} \frac{3^x-3}{2^x-2} = \lim_{x\to 1} \frac{(\ln 3)\,3^x}{(\ln 2)\,2^x} = \frac{3\ln 3}{2\ln 2}$$

$\boxed{4}$

$$= \frac{\ln 27}{\ln 4}$$

ln We need to consider

$$\ln\left((1+x)^{1/x}\right) = \frac{1}{x}\ln(1+x)$$

$$= \frac{\ln(1+x)}{x}$$

Now

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$$

So

$$\lim_{x\to 0} \ln\left(\underbrace{(1+x)^{1/x}}\right) = 1$$

Therefore $\lim_{x\to 0} (1+x)^{1/x} = e.$