MATH 125 Examination 1 January 27, 2009 Name Key

For a full credit give your answers as exact numbers, not decimal approximations.

1. Calculate the average value of the following two functions.

   (a) The function $f(x) = |x| + 1$ over the interval $[-1, 2]$.

   (b) The function $g(x) = \lfloor x \rfloor = \text{floor}(x)$ over the interval $[e, \pi]$.

2. The graph of some function f is given in Figure 1 below.

   Consider the following four numbers.

   n1: The average value of $f(x)$ on $0 \le x \le a$.

   n2: $\int_0^a f(x)\,dx$. **n3:** $\int_0^a f'(x)\,dx$.

   n4: The average value of the rate of change of $f(x)$ on $0 \le x \le a$. (That is the average value of $f'(x)$ on $0 \le x \le a$.)

   Answer the following questions.

   (a) For each of the four numbers introduced above, **explain how** it can be visualized on the figure. **State clearly** whether a number is represented by a length (horizontal or vertical), slope or an area.

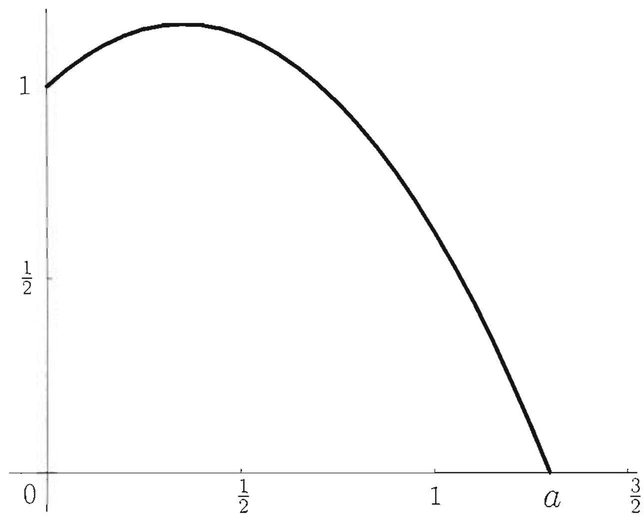
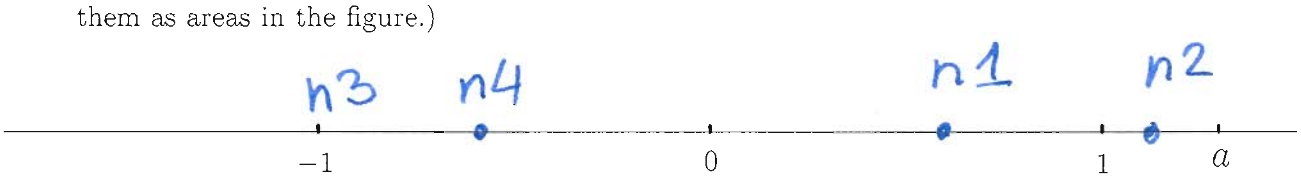


Figure 1: Compare the numbers listed

   (b) Place the symbols **n1**, **n2**, **n3**, **n4**, approximately where the corresponding numbers are located on the axis below. (Hint: In order to compare the numbers a and 1 to the other numbers visualize them as areas in the figure.)



3. Let $x > 0$. Consider the curves $y = \sqrt{x}$, $y = \dfrac{1}{\sqrt{x}}$ and the vertical line $x = 4$.

   (a) Find the exact area inclosed by the given curves and the vertical line.

   (b) Give a simple argument why this area must be greater than $\dfrac{9}{4}$.

4. Let $t \ge 0$. Consider the function $G(t) = \displaystyle\int_{\sqrt{t}}^{\sqrt{2t}} e^{-x^2}\,dx$.

   (a) (i) Calculate $G(0)$. Guess $\lim_{t\to\infty} G(t)$. Provide a short explanation of your guess.

   (ii) Which of the following is true and why?

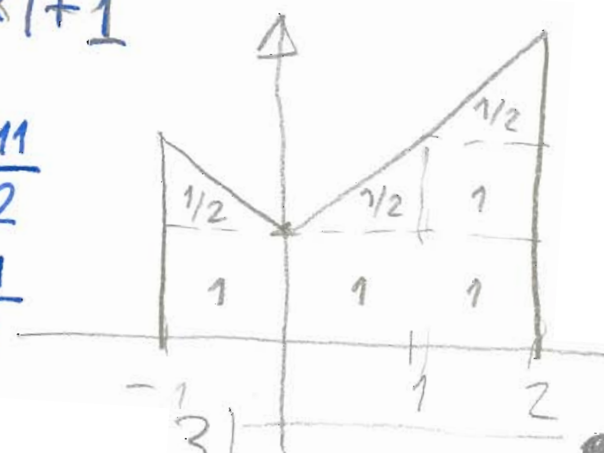
   $\boxed{G(t) < 0 \text{ for all } t > 0.}$ $\boxed{G(t) > 0 \text{ for all } t > 0.}$ $\boxed{\text{Neither of the previous two.}}$

   (b) Calculate $G'(t)$. Simplify your answer.

   (c) Does the function $G(t)$ have a global maximum, global minimum, or neither? Explain.

$\boxed{1}$

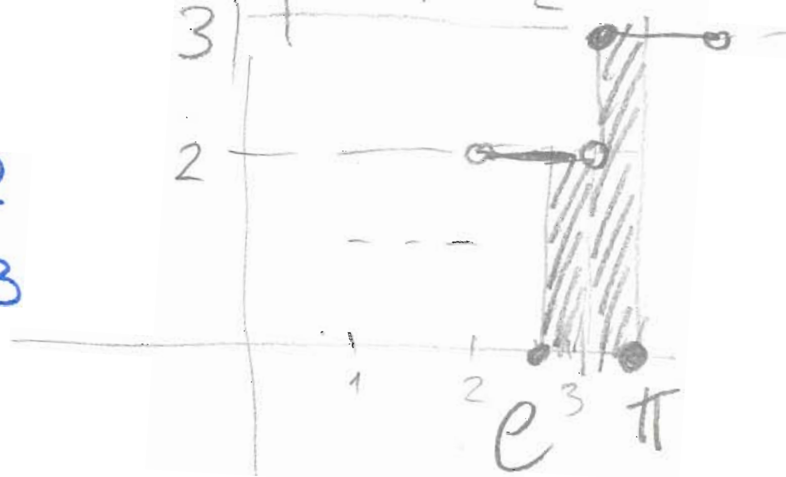
① ⓐ $f(x) = |x| + 1$



$\text{area} = 5\tfrac{1}{2} = \frac{11}{2}$

$\text{averag} = \frac{11/2}{3} = \frac{11}{6}$

ⓑ $g(x) = \lfloor x \rfloor$



area $(3-e)*2$
$+(\pi-3)*3$

$= 6 - 2e + 3\pi - 9$

$= 3\pi - 2e - 3$

average $\frac{3\pi - 2e - 3}{\pi - e}$

② n1 vertical ~~length~~ length; height of the rectangle with the same area as $\int_0^a f(x)\,dx$

n2 the area under the graph.

n3 vertical length, it is negative here
$f(a) - f(0) = -1$

n4 slope joining $(0,1)$ and $(a,0)$
$-\frac{1}{a}$.

③ ⓐ

$$\boxed{2}$$

$$\int_1^4 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx =$$

$$= F(4) - F(1)$$

where:

$$F(x) = \frac{2}{3}x^{3/2} - 2\sqrt{x}$$

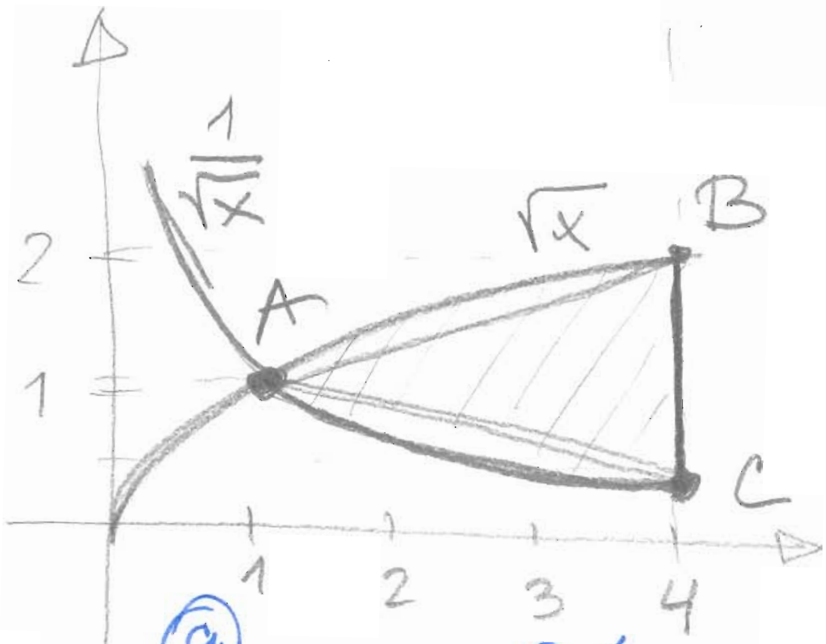
$$F(4) = \frac{2}{3} \cdot 2^3 - 2 \cdot 2$$

$$= \frac{16}{3} - \frac{12}{4}$$

$$= 4/3$$

$$F(1) = \frac{2}{3} - 2 = -\frac{4}{3}$$

$$\boxed{F(4) - F(1) = \frac{8}{3}}$$



ⓐ Area $8/3$

ⓑ The area of the ΔABC is

$$\frac{3 * \frac{3}{2}}{2} = \frac{9}{4}$$

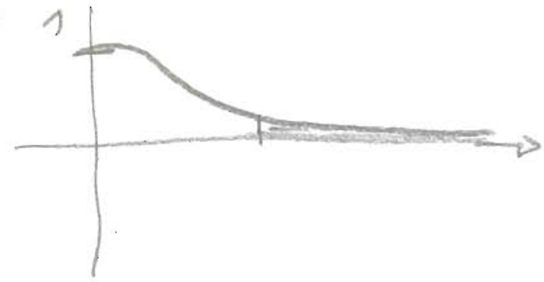
so $\frac{8}{3} > 9/4$

what is TRUE!

4 a $G(0) = \int_0^0 e^{-x^2}\,dx = 0$

i $\lim_{t\to\infty} G(t) = 0$. Why?

The function e^{-x^2} is very, very small for large x



so $\int_{\sqrt{t}}^{\sqrt{2t}} e^{-x^2}\,dx \le \underbrace{e^{-t}\left(\sqrt{2t}-\sqrt{t}\right)}_{\text{very small for large } t.}$

ii $\boxed{G(t) > 0}$ for all $t > 0$ since the function e^{-x^2} is always positive.

b $G'(t) = e^{-2t} * \frac{1}{\cancel{2}}\frac{1}{\sqrt{2t}} * \cancel{2} - e^{-t}\frac{1}{2\sqrt{t}}$

$$= \frac{e^{-2t}}{\sqrt{2t}} - \frac{e^{-t}}{2\sqrt{t}} =$$

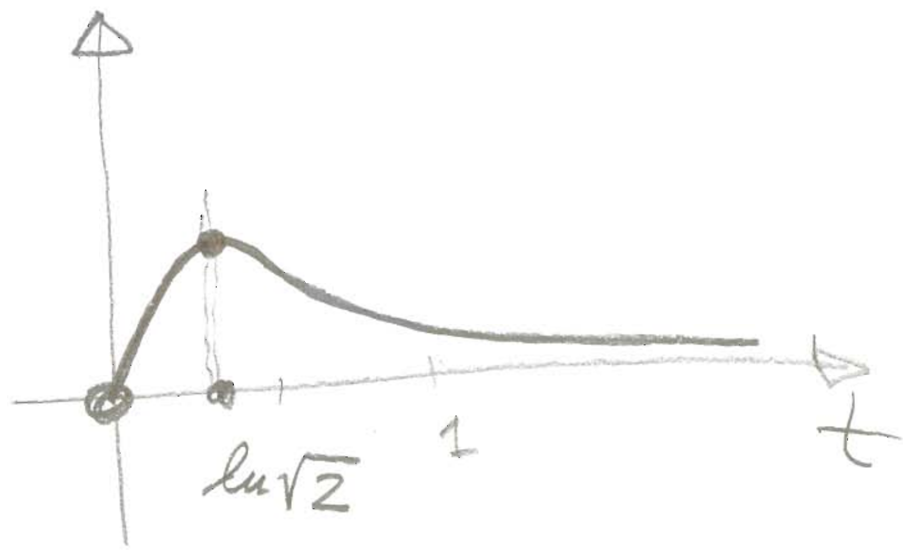
$$= \frac{e^{-2t}}{\sqrt{2t}}\left(1 - \frac{e^{t}}{\sqrt{2}}\right)$$

$$= \frac{e^{-2t}}{2\sqrt{t}}\left(\sqrt{2} - e^{t}\right)$$

(c) We see that

$$G'(t) > 0 \text{ for } t < \ln\sqrt{2}$$
$$G'(t) = 0 \text{ for } t = \ln\sqrt{2}$$
$$G'(t) < 0 \text{ for } t > \ln\sqrt{2}.$$

Thus $G(t)$ looks like



It has a ~~local~~ global maximum

at $t = \ln\sqrt{2} = \frac{1}{2}\ln 2$.