MATH 204 Examination 1 January 29, 2013 Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
There are four problems. Each is worth 25 points.

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 3 & 5 & 2 \end{bmatrix}$.

   (a) Can the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ be written as a linear combination of the columns of A? If your answer is yes, then write $\mathbf{b}$ as a linear combination of the columns of A.

   (b) Do the columns of A span $\mathbb{R}^3$? If your answer is no, then provide one vector in $\mathbb{R}^3$ which cannot be written as a linear combination of the columns of A.

2. Consider the homogenous equation $A\mathbf{x} = \mathbf{0}$. It is given that the matrix A is row equivalent to

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   Write the general solution of the equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

3. (a) Write the solution set of the equation $x_1 - 2x_2 + 3x_3 = 0$ in parametric vector form.

   (b) Write the solution set of the equation $x_1 - 2x_2 + 3x_3 = 4$ in parametric vector form.

   (c) Notice that the equations in (3a) and (3b) have identical left hand sides. Provide a geometric comparison of the solution sets in (3a) and (3b).

4. (a) Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ h \end{bmatrix}$. Determine for what values of the parameter h in $\mathbf{v}_3$ the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

   (b) Determine if the columns of the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 5 & 2 \\ 3 & 2 & 5 \\ 1 & 2 & -1 \end{bmatrix}$ form a linearly independent set.

① $\left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 3 & 5 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\boxed{1}$

ⓐ Yes, $\vec{b}$ is a linear combination of the columns of A. For example,

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \checkmark$$

ⓑ I need to check:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 1 & 1 & 0 & b_2 \\ 3 & 5 & 2 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - 3b_1 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right]$$

If $b_3 - b_2 - 2b_1 \neq 0$ the vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is <u>Not</u> in the span of col. of A. For example $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is NOT in the span of col. of A.

② A is 4×5 matrix. there are 5 unknowns: x_1, x_3, x_4 are basic, x_2 and x_5 are free. Set $x_2 = s, x_5 = t$. Then the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s+3t \\ s \\ -t \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}.$$

3 a $x_1 = 2x_2 - 3x_3$

$x_2 = s$, $x_3 = t$ are free variables.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2s - 3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

b $x_1 = 4 + 2x_2 - 3x_3$

$x_2 = s$, $x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 + 2s - 3t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

c Geometrically both solutions are planes in $\mathbb{R}^3$. These planes are parallel: the first plane passes through the origin, the second plane passes through the head of $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$.

4 a $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & h \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & -3 & h-9 \end{bmatrix}$ $\boxed{3}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3/2 \\ 0 & 1 & 3-\frac{h}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3/2 \\ 0 & 0 & 3/2-\frac{h}{3} \end{bmatrix}$$

For $h = 9/2$ the system $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ will have a nontrivial solution. such ~~that~~ solution is for example:

$x_3 = \frac{2}{3}$, $x_2 = -1$, $x_1 = 0$.

$$0\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 3 \\ 3 \\ 9/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 5 & 2 \\ 3 & 2 & 5 \\ 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 6 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The columns of A do NOT form a lin. ind. set. They are linearly dependent.

$$1 * (\text{col } 1) + 2(\text{col } 2) - 3(\text{col } 3) = \vec{0}$$