MATH 204 Examination 2 February 19, 2013 Name ______________________

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
There are four problems. Each is worth 25 points.

1. (a) A linear transformation $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ reflects points through the x_1-axes. Find the standard matrix of this linear transformation.
   (b) A linear transformation $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ reflects points through the line $x_2 = -x_1$. Find the standard matrix of this linear transformation.
   (c) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_1-axes and then reflects points through the line $x_2 = -x_1$. Find the standard matrix of this linear transformation.
   (d) Show that T also can be described as a linear transforation that rotates points about the origin. What is the angle of that rotation?

2. Let $B = \begin{bmatrix} 1 & -2 & 3 & 4 \\ 4 & 2 & -3 & 0 \\ -4 & 2 & 1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -3 & 6 \\ 1 & 2 & 4 \end{bmatrix}$.
   (a) Find the second column of BC or state and explain why it is not defined.
   (b) Find the second column of CB or state and explain why it is not defined.
   (c) Consider the matrices BB and CC. Only one of these matrices is defined. State which one and calculate the entry in the second row and the first column of that matrix.

3. In this problem A is $n \times n$ matrix. Without using the Invertible Matrix Theorem give explanations for the following implications.
   (a) If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then there is an $n \times n$ matrix D such that $AD = I$. Explain why.
   (b) If there is an $n \times n$ matrix D such that $AD = I$, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why.
   (c) If there is an $n \times n$ matrix C such that $CA = I$, then the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why.

4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.
   (a) Find A^{-1}. Prove that your answer is correct by calculating AA^{-1}.
   (b) Use the inverse A^{-1} to find x_1, x_2, x_3 such that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\boxed{1}$

(1a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(1b) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (1c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(1d) This is the rotation by $\frac{3\pi}{2}$ clockwise.
The general rotation matrix is
$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \frac{3\pi}{2}$ it is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(2) (a) The multiplication BC is NOT possible since B is 3×4 and C is 3×3

We need ~~m~~ $(m \times n)$ mult. by $n \times p$
same

(b) CB is 3×3 matrix. To find the second column we multiply each row of C by the ~~second~~ second colum of $\underline{B}$:

$2*(-2)+3*(2)+4*2 = 10$
$0\cdot(-2)+(-3)2+6*2 = 6$
$1*(-2)+2(2)+4*2 = 10$

(c) Only CC is defined.
2nd row, 1st col. $0*2+(-3)*0+6=6$

(3a) $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$. $\boxed{2}$

Then $A\vec{x} = \vec{e}_1$, $A\vec{x} = \vec{e}_2, \ldots, A\vec{x} = \vec{e}_n$ each has a solution. Call solutions $\vec{d}_1$, $\vec{d}_2$, $\vec{d}_n$

Then $D = [\vec{d}_1 \ \vec{d}_2 \ \ldots \ \vec{d}_n]$

satisfies $AD = I$.

(3b) Assume $AD = I$. Then $A(D\vec{b}) = \vec{b}$. Then the vector $D\vec{b}$ solves the equation $A\vec{x} = \vec{b}$.

(3c) Assume $CA = I$. Let $\vec{x}$ be such that $A\vec{x} = \vec{0}$. Then $(CA)\vec{x} = 0$. But $CA = I$ so $\vec{x} = \vec{0}$. Therefore $\vec{0}$ is the only solution

$\boxed{3}$

4

a

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

Verify $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ok!

b the solution is

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-2+2 \\ 1-1+2 \\ 1-2+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$