Fall 2014   Math 204   Topics for the final

1.1 Linear systems. Know:

- How to distinguish linear equations from nonlinear equations.
- What is the coefficient matrix and the augmented matrix of a linear system.
- What are equivalent linear systems and what are three basic operations which transform a linear system into an equivalent simpler system.
- The existence and uniqueness question for linear system.

1.2 Row reduction and row echelon forms. Know:

- The definitions of REF and RREF of a matrix and how to use the Row Reduction Algorithm to transform a matrix to REF and RREF.
- The concepts of a pivot position and a pivot column in a matrix and the connection with the basic and free variables of a system.
- How to use row reduction to find the general solution of a linear system and how to write this solution in parametric form.
- The Existence and Uniqueness Theorem.

1.3 Vector equations. Know:

- How to write a linear system as one vector equation.
- Algebraic operations in the vector space $\mathbb{R}^n$, their geometric illustrations in $\mathbb{R}^2$ and $\mathbb{R}^3$.
- The concepts of a linear combination of vectors and a span of vectors and the geometric interpretation of a span in $\mathbb{R}^2$ and $\mathbb{R}^3$.

1.4 The matrix equation $A\mathbf{x} = \mathbf{b}$. Know:

- The definition of matrix-vector product and its basic properties: $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, $A(c\mathbf{u}) = cA\mathbf{u}$ (Theorem 5).
- The matrix equation, the vector equation and the linear system which have the same solution set (Theorem 3).
- Four equivalent ways of saying: For every $\mathbf{b} \in \mathbb{R}^m$ the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

1.5 Solutions sets of linear systems. Know:

- The geometric illustration of the expression $\mathbf{p} + t\mathbf{v}$ where t is an arbitrary scalar and $\mathbf{p}$ and $\mathbf{v}$ are fixed vectors in $\mathbb{R}^n$.
- The geometric illustration of the expression $\mathbf{p} + s\mathbf{u} + t\mathbf{v}$ where t and s are arbitrary scalars and $\mathbf{p}$, $\mathbf{u}$ and $\mathbf{v}$ are fixed vectors in $\mathbb{R}^n$.
- How to write a solution of a linear system in parametric vector form.
- The relationship between the solution sets of a nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ and the corresponding homogeneous equation $A\mathbf{x} = \mathbf{0}$ (Theorem 6).

1.6 Applications of linear systems. Know:

- How to use linear system to find equilibrium prices for simple economic models consisting of two and three sectors of economy.
- How to use linear systems to balance chemical equations; in particular how to find smallest positive integers that balance a chemical equation.

1.7 Linear independence. Know:

- ➢ The definitions of linear independence and linear dependence and how to implement them to decide whether given vectors are linearly dependent or independent.
- ➢ The meaning of linear independence/dependence in the case of one vector and two vectors.
- ➢ Two simple sufficient conditions for the linear dependence (Theorems 8 and 9).
- ➢ Characterization of linearly dependent sets (Theorems 7).

1.8 Linear transformations. Know:

- ➢ That in this context the words transformation, mapping and function are synonyms.
- ➢ The definition of a **linear transformation**. Let n and m be positive integers. A transformation T defined on $\mathbb{R}^n$ and with the values in $\mathbb{R}^m$ is **linear** if the following two conditions are satisfied: $T(\mathbf{x}+\mathbf{y}) = T\mathbf{x} + T\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $T(c\mathbf{x}) = cT\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$.
- ➢ How to associate pictures to formulas and formulas to pictures.

1.9 Matrix of a linear transformations. Know:

- ➢ The most important theorem on the **standard matrix for a linear transformation**: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists a unique $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. In fact, the jth column of A is $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix.
- ➢ How to use the above theorem to get the standard matrix for a rotation in $\mathbb{R}^2$ (which is a linear transformation of $\mathbb{R}^2$).
- ➢ How to use the above theorem to get the standard matrices of transformations in Tables 1, 2, 3, 4 in Section 1.9. Also Exercises 1-22 all use this idea.
- ➢ Definitions of **one-to-one** and **onto** for transformations and the characterizations in Theorems 11 and 12. Exercises 23-30.

2.1 Matrix operations. Know:

- ➢ How to add matrices and multiply matrices with a scalar and properties of these operations.
- ➢ How to multiply two matrices (the definition and row-column rule for the computation).
- ➢ Properties of matrix multiplication.
- ➢ The transpose of a matrix and its properties.

2.2 The inverse of a matrix. Know:

- ➢ The definition of an invertible matrix and the definition of an inverse of a matrix.
- ➢ The easy inverses: 2×2 matrices, elementary matrices, product of invertible matrices.
- ➢ An algorithm for finding A^{-1} and its connection with elementary matrices, posts of May 4 and May 5.
- ➢ How to use inverse to solve the equation $A\mathbf{x} = \mathbf{b}$.
- ➢ Theorem 7 and its proof.

2.3 Characterization of invertible matrices. Know:

- ➢ The statement and the proof of the invertible matrix theorem. (See examples of the proofs that we did in class.)

3.1 Determinants. Know:

- ➢ The definition and the properties (Theorem 1, Theorem 2) of determinants and how to use them to calculate determinants.

3.2 Properties of determinants. Know:

- ➢ The properties: how row operations change determinant and how to use this property to calculate determinants.
- ➢ A square matrix is invertible if and only if $\det A \neq 0$.
- ➢ The multiplicative property of determinants ($\det(AB) = (\det A)(\det B)$ and how to use it to solve problems (Exercises 29, 31).
- ➢ The linearity property of the determinant function (page 197) and how to use it to calculate determinants.
- ➢ $\det A^T = \det A$.

3.3 Volume. Know:

- ➢ If A is 2×2 matrix, then the area of the parallelogram determined by the columns of A is $|\det(A)|$. If A is 3×3 matrix, then the volume of the parallelepiped determined by the columns of A is $|\det(A)|$.

4.1 Vector spaces and subspaces. Know:

- ➢ The concept of an abstract vector space; ten axioms: AE (addition exist), AA (addition is associative), AC (addition is commutative), AZ (addition has zero), AO (addition has opposites), SE (scaling exists), SA (scaling is "associative"), SO (scaling with one), DL (distributive law), DL (distributive law)
- ➢ The concept of a subspace; three defining properties of a subspace $\mathcal{U}$: **(1)** $0 \in \mathcal{U}$, **(2)** $u+v \in \mathcal{U}$ whenever $u, v \in \mathcal{U}$, **(3)** $\alpha u \in \mathcal{U}$ whenever $u \in \mathcal{U}$ and $\alpha \in \mathbb{R}$
- ➢ The concept of a span
- ➢ Examples of vector spaces and their subspaces: vector spaces of polynomials and vector spaces of functions

4.2 Null spaces, column spaces and linear transformations. Know:

- ➢ Null space: the definition, the proof that it is a subspace, how to find a null space of a given matrix, how to write it as a span of vectors, and how to find its basis (this is explained in 4.3)
- ➢ Column space: the definition, how to decide whether a given vector is in the column space of a given matrix, and how to find its basis (this is explained in 4.3)
- ➢ The importance of equalities $\operatorname{Nul} A = \{\vec{0}\}$ and $\operatorname{Col} A = \mathbb{R}^m$ for a given matrix $m\times n$ matrix A
- ➢ The definitions of kernel and range of a linear transformation. Exercises 31, 32, 33

4.3 Linearly independent sets: Bases. Know:

- ➢ The definition of linearly independent and linearly dependent vectors; the characterization of linearly dependent sets in Theorem 4
- ➢ The definition of a basis of a vector space
- ➢ The spanning set theorem
- ➢ How to find bases for $\operatorname{Nul} A$ and $\operatorname{Col} A$ for a given $m\times n$ matrix A
- ➢ Exercise 23, 24, 34

4.4 Coordinate systems. Know:

- ➢ The unique representation theorem, the definition of a coordinate mapping, the meaning of the symbol $[\mathbf{v}]_{\mathcal{B}}$ for a given vector $\mathbf{v}$ and a basis $\mathcal{B}$
- ➢ The importance of the matrix

$$P_{\mathcal{B}} = \begin{bmatrix} \vec{b}_1 \cdots \vec{b}_n \end{bmatrix}$$

for a given basis $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ for $\mathbb{R}^n$ (this is a special change-of-coordinate matrix, more in Section 4.7)

- Theorem 8: Given a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of a vector space $\mathcal{V}$, the coordinate mapping $\mathbf{v} \mapsto [\mathbf{v}]_{\mathcal{B}}$ is a one-to-one linear transformation from $\mathcal{V}$ onto $\mathbb{R}^n$.
- The coordinate mapping for polynomials, Examples 5 and 6
- Exercises 10, 11, 13

4.5 The dimension of a vector space. Know:

- Theorem 9: Let p and n be positive integers. Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space $\mathcal{V}$. Let $\mathbf{v}_1, \dots, \mathbf{v}_p$ be vectors in $\mathcal{V}$. If $p > n$, then the vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$ are linearly dependent.
- A proof of Theorem 9 and the statement of the contrapositive of Theorem 9
- Theorem 10: If a vector space $\mathcal{V}$ has a basis of n vectors, then every basis of $\mathcal{V}$ must consist of n vectors.
- The definition of finite dimensional vector space and the definition of the dimension of a finite dimensional vector space
- Theorem 11 and Theorem 12
- How to determine dimensions of $\operatorname{Nul} A$ and $\operatorname{Col} A$ for a given matrix $m \times n$ matrix A
- Exercise 23

4.6 Rank. Know:

- The concept of a row space, $\operatorname{Row} A$, of a given matrix A
- Theorem 13: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in row echelon form, then the nonzero rows of B form a basis for the row space of A (which is the same as the row space of B).
- The definition of the rank of a matrix
- The nullity of a matrix A is the dimension of $\operatorname{Nul} A$
- That the Rank Theorem in the book is more often called the Rank-nullity theorem. This theorem has three important claims.
- **The Rank-Nullity Theorem**: **(1)** The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. **(2)** This common dimension, the rank of A, also equals the number of pivot positions in A. **(3)** The rank of A and the dimension of $\operatorname{Nul} A$ add up to the number of columns of A. That is
$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$
- Both $\operatorname{Nul} A$ and $\operatorname{Row} A$ are subspaces of $\mathbb{R}^n$. The only vector which is in both $\operatorname{Nul} A$ and $\operatorname{Row} A$ is the zero vector. A union of a basis for $\operatorname{Nul} A$ and a basis for $\operatorname{Row} A$ is a basis for $\mathbb{R}^n$.
- Four fundamental subspaces determined by A and relationships among their dimensions.
- Exercises 27-30 among others

4.7 Change of bases (in fact: Change of coordinates). Know:

- That the matrix M with the property $[\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$**. It is denoted $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}$ and it is calculated as
$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$
- $\left(\underset{\mathcal{C} \leftarrow \mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B} \leftarrow \mathcal{C}}{P}$
- That there is a special basis of $\mathbb{R}^n$, called the standard basis, which consists of the columns of the identify matrix I_n. It is denoted by $\mathcal{E}$.

- That for a basis $\mathcal{B} = \left\{\vec{b}_1, \ldots, \vec{b}_n\right\}$ for $\mathbb{R}^n$ we have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \left[\,\vec{b}_1 \cdots \vec{b}_n\,\right] = P_{\mathcal{B}}$$

  (the matrix $P_{\mathcal{B}}$ was introduced in Section 4.4).
- $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\right)\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right) = \left(\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}\right)^{-1}\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \left[\,\vec{c}_1 \cdots \vec{c}_n\,\right]^{-1}\left[\,\vec{b}_1 \cdots \vec{b}_n\,\right]$
  where $\mathcal{C} = \left\{\vec{c}_1, \ldots, \vec{c}_n\right\}$ is another basis for $\mathbb{R}^n$
- Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues. Know:

- The definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.
- The definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.
- That the eigenvalues of a triangular matrix are the entries on its main diagonal.
- **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in more formal mathematical language: Let A be an $n\times n$ matrix, let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}$. If $A\vec{v}_k = \lambda_k\vec{v}_k$, $\vec{v}_k \neq \vec{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \ldots, m$, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly independent.
- A proof of the above theorem for $m = 2$ vectors.

5.2 The characteristic equation. Know:

- That λ is an eigenvalue of an $n\times n$ matrix A if and only if $\det(A - \lambda I) = 0$
- How to calculate $\det(A - \lambda I)$ (this is the characteristic equation) for 2×2 and 3×3 matrices
- Application to dynamical systems

5.3 Diagonalization. Know:

- **Theorem.** (The diagonalization theorem) An $n\times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors
- How to decide whether a given 2×2 and 3×3 matrix A, is diagonalizable or not; if it is diagonalizable, how to find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.