MATH 204 Examination October 30, 2015 Name ______________________

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
On in class exams I assign four problems. Each is worth 25 points.
I try to assign problems from different topics that we covered.
Below are several problems to help you get used to my style of exam questions.

1. In this problem A is $n \times n$ matrix. Without using the Invertible Matrix Theorem give explanations for the following implications.

   (a) If the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$, then there is an $n \times n$ matrix D such that $AD = I$. Explain why.

   (b) If there is an $n \times n$ matrix D such that $AD = I$, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why.

   (c) If there is an $n \times n$ matrix C such that $CA = I$, then the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why.

2. This problem is about invertible matrices. Let A be an $n \times n$ matrix.

   (a) State the definition of an invertible matrix.

   (b) Prove the implication: If A is invertible, then A is row equivalent to I_n.

   (c) Prove the implication: If A is row equivalent to I_n, then A is invertible.

3. For each matrix below determine whether it is invertible or not. Explain your claim. If a matrix is invertible find its inverse.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & -1/2 \\ -2 & 1 \end{bmatrix}.$$

4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

   (a) Find A^{-1}. Prove that your answer is correct by calculating AA^{-1}.

   (b) Use the inverse A^{-1} to find x_1, x_2, x_3 such that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

5. Determine h such that

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 1 & h \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0.$$

6. Let a, b, c, d be real numbers. Calculate the following three determinants:

$$\begin{vmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{vmatrix}, \quad \begin{vmatrix} a & 0 & 0 & 0 & b \\ 0 & a & 0 & b & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & c & 0 & d & 0 \\ c & 0 & 0 & 0 & d \end{vmatrix}, \quad \begin{vmatrix} a & 0 & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & 0 & b & 0 \\ 0 & 0 & a & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & d & 0 & 0 \\ 0 & c & 0 & 0 & 0 & d & 0 \\ c & 0 & 0 & 0 & 0 & 0 & d \end{vmatrix}.$$

7. Consider the matrices $A = \begin{bmatrix} 1 & -3 \\ -1 & k \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$

   (a) What values of k (if any) will make A invertible?

   (b) What values of k (if any) will make AB invertible?

   (c) What values of k (if any) will make $AB = BA$?

8. Let A be an unknown square matrix. To A we perform the following row operations:

   - Row 1 gets replaced by $\frac{1}{2}$ times Row 1. ($R_1 \mapsto \frac{1}{2}R_1$)
   - Rows 2 and 3 are swapped ($R_2 \mapsto R_3, R_3 \mapsto R_2$)
   - Row 2 gets replaced by -3 Row 2. ($R_2 \mapsto -3R_2$)
   - Row 3 gets replaced by Row 3 minus 6 Row 2. ($R_3 \mapsto R_3 - 6R_2$)

   The resulting matrix B has determinant $\det B = 4$. What is the determinant of the unknown matrix A.

9. Let A be an unknown 3×3 matrix. To A we perform the following row operations:

   - Row 1 gets replaced by $\frac{1}{2}$ times Row 1. ($R_1 \mapsto \frac{1}{2}R_1$)
   - Rows 2 and 3 are swapped ($R_2 \mapsto R_3, R_3 \mapsto R_2$)
   - Row 2 gets replaced by -3 Row 2. ($R_2 \mapsto -3R_2$)
   - Row 3 gets replaced by Row 3 minus 6 Row 2. ($R_3 \mapsto R_3 - 6R_2$)

   The resulting matrix B is the identity matrix. What is the matrix A?

10. Consider the matrix $V = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix}$

    (a) Calculate the determinant $\det V$.

(b) Find values for x, y, z such that V is invertible and values for x, y, z in which V is not invertible.

11. Determine whether it is possible to write the matrix $M = \begin{bmatrix} 3 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ as a product of elementary matrices. If you claim that it is possible to write M as a product of elementary matrices, then find elementary matrices whose product is M. If you claim that it is not possible to write M as a product of elementary matrices, then justify your claim.

12. Calculate the determinant $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 0 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{vmatrix}$.