MATH 204 Examination December 3, 2015

Name ______________________

Give detailed explanations for your answers.
On a final exams I assign eight problems. Each is worth 12.5 points.
I try to assign problems from different topics that we covered.
It is possible that three out of eight problems would be from Chapter 5. Below are several problems to help you get used to my style of exam questions.

1. Let $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$.

   (a) Find a basis of $\mathbb{R}^2$ which consists of eigenvectors of A.

   (b) Write $\vec{x}_0$ as a linear combination of the basis vectors found in 1a.

   (c) Let k be a positive integer. Write $A^k\vec{x}_0$ as a linear combination of the basis vectors found in 1a.

   (d) What can you say about a vector $A^k\vec{x}_0$ for large positive integer k, say $k = 1000$?

2. Consider the matrix $A = \begin{bmatrix} 5 & h & -2 & 1 \\ 0 & 3 & h & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find the value(s) of $h \in \mathbb{R}$ such that the matrix A is diagonalizable.

3. Consider the matrix $A = \begin{bmatrix} 0 & -5 & 3 \\ 1 & 6 & -3 \\ 3 & 9 & -4 \end{bmatrix}$.

   (a) Diagonalize the matrix A.

   (b) Calculate A^6 without using your calculator.

   (c) Calculate A^k, where k is an even integer.

4. Is the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ diagonalizable?

5. Consider the matrix $A = \begin{bmatrix} 1 & -3 & -3 \\ 3 & 7 & 3 \\ -3 & -3 & 1 \end{bmatrix}$. The eigenvalues of this matrix are 1 and 4.

   (a) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

   (b) Let D be the diagonal matrix found in 5a. Find eight distinct diagonal matrices $D_1, \dots, D_8$ such that $(D_j)^2 = D$.

   (c) We have seen how we can use the expression $A = PDP^{-1}$ to easily compute high powers of the matrix A. Use this idea and 5b to find several matrices B such that $B^2 = A$.

6. For a real number x consider the matrix $A_x = \begin{bmatrix} 1 & x & 0 \\ x & x & x \\ 0 & x & 1 \end{bmatrix}$.

(a) Prove that $\lambda = 1$ is an eigenvalue of the matrix A_x for all real numbers x. Find a corresponding eigenvector.

(b) Does there exist an $x \in \mathbb{R}$ such that the eigenspace corresponding to the eigenvalue $\lambda = 1$ is two dimensional? Find all such x.

(c) Prove that all the eigenvalues of A_x are real.

(d) For which value(s) of x is $\lambda = -1$ an eigenvalue of A_x?

(e) For which value(s) of x all eigenvalues of A_x are positive?

(f) For which value of x the distance between the smallest and the largest eigenvalue of A_x is the smallest possible.

(g) Is the matrix A_x diagonalizable for every $x \in \mathbb{R}$?

THIS IS THE ORDER OF DIFFICULTY: 6d, 6a, 6b, 6c, 6g, 6e, 6f.

7. Repeat the above problem for the matrix $A_x = \begin{bmatrix} 0 & x & 1 \\ x & x & x \\ 1 & x & 0 \end{bmatrix}$. Make one change: swap the expressions $\lambda = 1$ and $\lambda = -1$.

THE LAST TWO PROBLEMS ARE TOO LONG FOR AN EXAM. I JUST WANTED TO MAKE A RECORD OF ALL INTERESTING QUESTIONS THAT CAME TO MIND.