MATH 204  Examination
October 14, 2019                          Name _____

1. Consider the following nonhomogeneous linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 3 \\ -x_1 \qquad\quad + x_3 + 2x_4 &= -1 \\ 2x_1 + x_2 \qquad\quad - x_4 &= 3 \end{aligned}$$

(a) Write the augmented matrix of this system and row reduce it to the reduced row echelon form.

(b) Write the solution of the given system in parametric vector form.

(c) Without doing any additional row reduction write the solution of the corresponding homogeneous equation in parametric vector form.

2. Let $A$ be a $4 \times 3$ matrix and let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Suppose that the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution. What can you say about linear dependence/independence of the columns of $A$? Justify your answer.

(b) Suppose that the matrix equation $A\mathbf{x} = \mathbf{b}$ has at least two solutions. What can you say about linear dependence/independence of the columns of $A$? Justify your answer.

3. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 3 \\ 2 & 1 & 3 \end{bmatrix}$.

(a) Can the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ be written as a linear combination of the columns of $A$? If your answer is yes, then write $\mathbf{b}$ as a linear combination of the columns of $A$.

(b) Do the columns of $A$ span $\mathbb{R}^3$? If your answer is no, then provide one specific vector in $\mathbb{R}^3$ which cannot be written as a linear combination of the columns of $A$.

4. It is given that the matrix $A$ is row equivalent to

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Write the general solution of the equation $A\mathbf{x} = \mathbf{0}$ in parametric vector form.

(b) Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b}$ in $\mathbb{R}^4$? Explain your answer.

5. (a) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Complete the sentence: The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if ....

(b) Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 8 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 7 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} h \\ 5 \\ 4 \end{bmatrix}$. Determine for what values of the parameter $h$ in $\mathbf{v}_3$ the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

(c) Determine if the columns of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ are linearly independent.

6. (a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the $x_2$-axes and then reflects points through the line $x_2 = x_1$. Find the standard matrix of this linear transformation.

(b) Show that $T$ also can be described as a linear transforation that rotates points about the origin. What is the angle of that rotation?

7. Consider the matrix
$$A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & c & 1 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

(a) For each $c$ calculate the reduced row echelon form of $A$. (Hint: there are only two distinct cases.)

(b) In each case above state whether the columns of $A$ span $\mathbb{R}^3$.

8. I considered a matrix equation $A\mathbf{x} = \mathbf{0}$ with a $3 \times 4$ matrix $A$. I found that the vector parametric form of the solution of the given vector equation is
$$\mathbf{x} = s \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$
where $s$ and $t$ are any numbers. Find the reduced row echelon form of the matrix $A$.