MATH 204 Examination December 8, 2019

Name ____________________

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
On a final exams I assign eight problems. Each is worth 12.5 points.
I try to assign problems from different topics that we covered.
Two out of eight problems may be from Chapter 5.

1. Let $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$.
   (a) Find a basis of $\mathbb{R}^2$ which consists of eigenvectors of A.
   (b) Write $\mathbf{x}_0$ as a linear combination of the basis vectors found in 1a.
   (c) Let k be a positive integer. Write $A^k\mathbf{x}_0$ as a linear combination of the basis vectors found in 1a.
   (d) What can you say about a vector $A^k\mathbf{x}_0$ for large positive integer k, say $k = 1000$?

2. Consider the matrix $A = \begin{bmatrix} 5 & h & -2 & 1 \\ 0 & 3 & h & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find the value(s) of $h \in \mathbb{R}$ such that the matrix A is diagonalizable.

3. Consider the matrix $A = \begin{bmatrix} 0 & -5 & 3 \\ 1 & 6 & -3 \\ 3 & 9 & -4 \end{bmatrix}$.
   (a) Diagonalize the matrix A.
   (b) Calculate A^6 without using your calculator.
   (c) Calculate A^k, where k is an even integer.

4. Is the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ diagonalizable?

5. Consider the matrix $A = \begin{bmatrix} 1 & -3 & -3 \\ 3 & 7 & 3 \\ -3 & -3 & 1 \end{bmatrix}$. The eigenvalues of this matrix are 1 and 4.
   (a) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.
   (b) Let D be the diagonal matrix found in 5a. Find eight distinct diagonal matrices $D_1, \dots, D_8$ such that $(D_j)^2 = D$.
   (c) We have seen how we can use the expression $A = PDP^{-1}$ to easily compute high powers of the matrix A. Use this idea and 5b to find several matrices B such that $B^2 = A$.

6. For a real number x consider the matrix $A_x = \begin{bmatrix} 1 & x & 0 \\ x & x & x \\ 0 & x & 1 \end{bmatrix}$.

(a) Prove that $\lambda = 1$ is an eigenvalue of the matrix A_x for all real numbers x. Find a corresponding eigenvector.

(b) Does there exist an $x \in \mathbb{R}$ such that the eigenspace corresponding to the eigenvalue $\lambda = 1$ is two dimensional? Find all such x.

(c) Prove that all the eigenvalues of A_x are real.

(d) For which value(s) of x is $\lambda = -1$ an eigenvalue of A_x?

(e) For which value(s) of x all eigenvalues of A_x are positive?

(f) For which value of x the distance between the smallest and the largest eigenvalue of A_x is the smallest possible.

(g) Is the matrix A_x diagonalizable for every $x \in \mathbb{R}$?

THIS IS THE INCREASING ORDER OF DIFFICULTY: 6d, 6a, 6b, 6c, 6g, 6e, 6f.

7. Repeat the above problem for the matrix $A_x = \begin{bmatrix} 0 & x & 1 \\ x & x & x \\ 1 & x & 0 \end{bmatrix}$. Make one change: swap the expressions $\lambda = 1$ and $\lambda = -1$.

THE LAST TWO PROBLEMS ARE TOO LONG FOR AN EXAM. I JUST WANTED TO MAKE A RECORD OF ALL INTERESTING QUESTIONS THAT CAME TO MIND.