MATH 204 Examination December 1, 2024 Name ______________________

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
On a final exams I assign eight problems. Each is worth $100/8 = 12.5$ points.
I try to assign problems from different topics that we covered.

Problem 1. Consider the matrix $A = \begin{bmatrix} 5 & h & -2 & 1 \\ 0 & 3 & h & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find the value(s) of $h \in \mathbb{R}$ such that the matrix A is diagonalizable.

Problem 2. Consider the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & -3 \end{bmatrix}$.

(a) Diagonalize the matrix A.

(b) Calculate A^6 without using your calculator.

(c) Calculate A^7 without using your calculator.

Problem 3. Is the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ diagonalizable? Rigorously justify your answer.

Problem 4. Consider the matrix $A = \begin{bmatrix} 1 & -3 & -3 \\ 3 & 7 & 3 \\ -3 & -3 & 1 \end{bmatrix}$. The eigenvalues of this matrix are 1 and 4.

(a) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

(b) Let D be the diagonal matrix found in a. Find eight distinct diagonal matrices $D_1, \ldots, D_8$ such that $(D_j)^2 = D$.

(c) We have seen how we can use the expression $A = PDP^{-1}$ to easily compute high powers of the matrix A. Use this idea and item (b) to find several matrices B such that $B^2 = A$.

Problem 5. Let

$$A = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}$$

be the matrix which tells us about the movement of a certain population between regions ONE and TWO in a unit time period. The number 0.7 tells us that 70% of the population in ONE stays in ONE, the number 0.4 tells us that 40% of the population in TWO moves to ONE, the number 0.3 tells us that 30% of the population in ONE moves to TWO, and the number 0.6 tells us that 60% of the population in TWO stays in TWO. Hence, if the current population in ONE is x_1 and the current population in TWO is x_2, then after one unit of time the populations will be

$$\begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.7x_1 + 0.4x_2 \\ 0.3x_1 + 0.6x_2 \end{bmatrix},$$

meaning that the population in ONE will be $0.7x_1 + 0.4x_2$ and the population in TWO will be $0.3x_1 + 0.6x_2$ after one unit of time. We assume that there is no other movement of the population except specified by the matrix A. We call $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ the population vector. The vectors

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A^2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A^3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \ldots, \quad A^k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \ldots$$

are population vectors after $1, 2, 3, \ldots, k, \ldots$ units of time.

(a) Find a basis of $\mathbb{R}^2$ which consists of eigenvectors of A.

(b) Assume that at the beginning the entire population is in region ONE. That is $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the population vector. Express this vector as a linear combination of the vectors found in (a).

(c) Let k be a positive integer. Give the formula for the population vector after k units of time, that is express the vector $A^k\mathbf{e}_1$ as a linear combination of the vectors in (a).

(d) What can you say about a vector $A^k\mathbf{e}_1$ for large positive integer k? In other words calculate

$$\lim_{k\to\infty} A^k\mathbf{e}_1.$$

(e) Calculate

$$\lim_{k\to\infty} A^k\mathbf{e}_2.$$

In this problem all the calculations must be done using the exact fractions, not approximate decimal numbers. View A as

$$A = \frac{1}{10}\begin{bmatrix} 7 & 4 \\ 3 & 6 \end{bmatrix}.$$

Problem 6. In this problem we consider the vector space $\mathbb{P}_2$ of all polynomials of degree at most 2. Recall that the standard basis for $\mathbb{P}_2$ is $\mathcal{S} = \{1, t, t^2\}$.

(a) Consider the polynomials $\mathbf{q}_1(t) = 1 - t^2, \mathbf{q}_2(t) = t, \mathbf{q}_3(t) = 1 + t^2$ in $\mathbb{P}_2$. Prove that $\mathcal{B} = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is a basis for $\mathbb{P}_2$.

(b) Find $\underset{\mathcal{B}\leftarrow\mathcal{S}}{P}$.

(c) Find the basis $\mathcal{C}$ for $\mathbb{P}_2$ if $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = A$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$.

Problem 7. Consider the polynomials $\mathbf{p}_1(t) = 2t^2 + 1$ and $\mathbf{p}_2(t) = t + 2$ in the vector space $\mathbb{P}_2$ of polynomials of degree at most 2. Find a polynomial $\mathbf{p}_3(t)$ (in $\mathbb{P}_2$) such that

$$\mathbb{P}_2 = \mathrm{Span}\{\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t)\}.$$

Prove that your answer is correct.

Problem 8. Let $\mathbb{R}^{2\times 2}$ denote the vector space of 2×2 matrices with the usual addition of matrices and multiplication of a matrix by a scalar. Consider the following set of four matrices

$$\mathcal{A} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

(a) Are matrices in $\mathcal{A}$ linearly independent? Justify your answer.

(b) Does the set $\mathcal{A}$ span $\mathbb{R}^{2\times 2}$? If so, prove it. If not, find a specific element of $\mathbb{R}^{2\times 2}$ which is not in the span of $\mathcal{A}$.

Problem 9. Denote by $\mathbb{R}^{2\times 2}$ the vector space of 2×2 matrices with the usual addition of matrices and multiplication of a matrix by a scalar. Let A be an element of the vector space $\mathbb{R}^{2\times 2}$. Decide which of the following are subspaces and justify your answer.

(a) $\mathcal{H} = \{X \in \mathbb{R}^{2\times 2} \mid XA = AX\}$.

(b) $\mathcal{K} = \{X \in \mathbb{R}^{2\times 2} \mid XA = 0\}$.

Problem 10. (a) Let $v_1, \ldots, v_m$ be vectors in a vector space $\mathcal{V}$. State the definition of linear independence for the vectors $v_1, \ldots, v_m$.

The rest of this problem is about the space $\mathbb{P}_2$ of all polynomials of degree at most 2.

(b) Prove that the polynomials $\mathbf{q}_0(x) = 1,\ \mathbf{q}_1(x) = x,\ \mathbf{q}_2(x) = x^2$ are linearly independent.

(c) Let $\mathcal{H}$ be the set of all polynomials $\mathbf{p}$ in $\mathbb{P}_2$ such that $\mathbf{p}(1) = 0$. Show that $\mathcal{H}$ is a subspace of $\mathbb{P}_2$.

(d) Find a basis for $\mathcal{H}$. Justify your answer.