MATH 209

Final Examination
March 15, 2011

Name Key

1. (a) Is it possible for the propositions $p \vee q$ and $\neg p \vee \neg q$ to be both false? Justify your answer.
   (b) Is it possible for the proposition $p \to (\neg p \to q)$ to be false? Justify your answer.
   (c) Prove or disprove: $((p \to q) \to r) \to ((r \to q) \to p)$ is a tautology.

2. The universe of discourse in this problem is the set of all integers. Consider the following three statements.

   (a) $\forall x\, \forall y\, (x^2 = y^2 \to x = y)$, (b) $\exists x\, \forall y\, (x^2 = y^2 \to x = y)$, (c) $\exists x\, \forall y\, (xy \geq x)$.

   Write the negation of each of these statements. Decide and state clearly which statements are true. Prove the statements which are true.

3. Let x and y be real numbers. Determine all possible values for $\lceil x + y \rceil$ in terms of $\lceil x \rceil$ and $\lceil y \rceil$. Illustrate all possible cases with some famous numbers (e.g., $\pi, e, \sqrt{2}, \sqrt{3}, \ldots$) as examples. Justify that all possible cases are included in your list.

4. (a) Let $S = \{a, b, c, d\}$. Define a specific bijection between the power set $P(S)$ and the set of all bit strings of length 4. (This bijection should be "logical" so that you can use it to answer (4c) below. Hint: $f(\emptyset) = 0000, f(S) = 1111$.)
   (b) What is the cardinality of the power set $P(S)$?
   (c) If a set has n elements, what is the cardinality of its power set? Prove your claim.

5. Let r be a real number such that $r \neq 0$ and $r \neq 1$. Let n be a nonnegative integer. State and prove the closed form expression formula for the geometric sum

   $$\sum_{j=0}^{n} r^j = 1 + r + \cdots + r^n.$$

   Hint: If you cannot remember this formula you might be able to guess it for $r = 2$ and $r = 1/2$. Then try to guess the general formula. If you do not succeed, then prove the formula for $r = 2$.

6. Define a sequence a_n recursively by: $a_0 = 1, \quad a_{n+1} = \sum_{j=0}^{n} a_j = a_0 + \cdots + a_n, \quad n \in \mathbb{N}$.
   (a) Compute $a_0, a_1, a_2, a_3, a_4, a_5$.
   (b) Use strong induction to prove that $a_n = 2^{n-1}$ for all positive integers n.

7. (a) How many bit strings of length 9 <u>do not contain</u> the pattern 00. (b) Based on the calculation in (a) count how many bit strings of length 9 contain at least one occurrence of the pattern 00 and at least one occurrence of 11. (Hints: (a) Place 1s first; how many; you decide. (b) Look at the complement; it is an inclusion-exclusion problem.)

8. (a) How many different strings can be made from the letters in REARRANGE, using all the letters?
   (b) How many ways are there to rearrange the letters in REARRANGE into two separate words? (such as: GREEN RAAR)

9. This is a "wallet" problem. Consider the equation $x_1 + x_2 + x_3 = 24$.
   (a) How many triples (x_1, x_2, x_3) of nonnegative integers satisfy the given equation ?
   (b) How many triples (x_1, x_2, x_3) of positive integers satisfy the given equation?
   (c) How many triples (x_1, x_2, x_3) of digits, that is $x_1, x_2, x_3 \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, satisfy the given equation? (There are not too many triples here; you can even count them all.)

10. Each student in a class of 28 chooses 14 other students in the class and sends each one an email. Prove that some pair of students must send each other emails.

1 (a) $p \vee q$ and $\neg p \vee \neg q$ cannot be both false. $p \vee q$ false implies p is false and q is false, ~~so~~ thus $\neg p$ and $\neg q$ are true so $\neg p \vee \neg q$ is true.

(b) $p \to (\neg p \to q)$ to be false we must have p T and $\neg p \to q$ F, but p T implies $\neg p$ is F so $\neg p \to q$ is always true. In other words $p \to (\neg p \to q)$ is a tautology.

(c) The given statement is not tautology since p F q T r T yields $(p \to q) \to r$ is T and $(r \to q) \to p$ F so the compound statement is false.

2 (a) neg. is $\exists x \exists y \, (x^2 = y^2 \wedge x \neq y)$
This is true $x = 1$, $y = -1$.

(b) $\forall x \exists y \, (x^2 = y^2 \wedge x \neq y)$ is the negation.
(b) is true set $x = 0$ then
$\forall y \quad 0 = y^2 \to y = 0$ is true.

(c) neg.: $\forall x \exists y \, (xy < x)$. (c) is true
set $x = 0$ then $\forall y \quad 0 \cdot y \geq 0$.

(3) $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$

if $\lceil x \rceil + \lceil y \rceil - 1 < x+y \leq \lceil x \rceil + \lceil y \rceil$

$\cancel{\lceil x \rceil + 1 =}$

$\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil - 1$

if $\lceil x \rceil + \lceil y \rceil - 2 < x+y \leq \lceil x \rceil + \lceil y \rceil - 1$

Note that always:

$$\lceil x \rceil - 1 < x \leq \lceil x \rceil$$

$$\lceil y \rceil - 1 < y \leq \lceil y \rceil$$

and thus $\lceil x \rceil + \lceil y \rceil - 2 < x+y \leq \lceil x \rceil + \lceil y \rceil$.

(4) (a)

$f(\emptyset) = 0000$
$f(\{a\}) = 1000$
$f(\{b\}) = 0100$
$f(\{c\}) = 0010$
$f(\{d\}) = 0001$
$f(\{a,b\}) = 1100$
$f(\{a,c\}) = 1010$
$f(\{a,d\}) = 1001$
$f(\{b,c\}) = 0110$
$f(\{b,d\}) = 0101$
$f(\{c,d\}) = 0011$
$f(\{a,b,c\}) = 1110$
$f(\{a,b,d\}) = 1101$
$f(\{a,c,d\}) = 1011$
$f(\{b,c,d\}) = 0111$
$f(\{a,b,c,d\}) = 1111$

All bitstrings used (onto)
distinct subsets
distinct bitstrings
It is a bijection!

2 b $|P(S)| = 2^4 = 16$

Note that there are 16 bit strings of length 4: this is by the product rule

$\square \ \square \ \square \ \square$

$2 * 2 * 2 * 2 = 16$

c If S has n elements then there exists a bijection between $P(S)$ and the set of bitstrings of length n.

The bijection is as in a. Let ~~be elements~~ all distinct elements of S.

$s_1, s_2, \dots, s_n$

If $A \subseteq S$, then

$f(A) = \square\square\square \cdots \square\square$

bit string

0 if $s_1 \notin A$, 1 if $s_1 \in A$

0 if $s_2 \notin A$, 1 if $s_2 \in A$

and so on ....

$f(\emptyset) = 00\dots0$ all zeros

$f(S) = 11\dots1$ all 1s.

5

$$S = 1 + r + \cdots + r^n$$

$$rS = r + r^2 + \cdots + r^{n+1}$$

$$rS - S = r^{n+1} - 1$$

$$S(r-1) = r^{n+1} - 1$$

$$S = \frac{r^{n+1} - 1}{r-1}$$

Example

$$1 + 2 + \cdots + 2^n = \frac{2^{n+1} - 1}{2-1}$$

$$= 2^{n+1} - 1$$

6 a

$a_0 = 1$ $\quad a_3 = 4$

$a_1 = 1$ $\quad a_4 = 8$

$a_2 = 2$ $\quad a_5 = 16, \ldots$

b $P(n)$: $a_n = 2^{n-1}$, $n \in \{1, 2, 3, \ldots\}$

BC $P(1)$ is true $a_1 = 2^{1-1} = 2^0 = 1$

IH assume $k \in \mathbb{Z}_+$ and $P(j)$ is true for all $j \in \{1, 2, \ldots, k\}$.

We need to prove $a_{k+1} = 2^k$.

5

By the definition of a_n we have

$$a_{k+1} = a_0 + a_1 + \cdots + a_k$$
$$\cancel{= 1 + 1 + \cdots + a_k}$$

Now use IH: $a_1 = 1, a_2 = 2, a_3 = 4$

$$a_j = 2^{j-1}, \; j = 1, 2, \ldots, k$$

$$a_{k+1} = \overset{a_0}{\downarrow}{1} + 1 + 2 + 4 + \cdots + 2^{k-1}$$

Problem 5 $= 1 + (2^k - 1) = 2^k = 2^{(k+1)-1}$

This Proves $P(k+1)$.

7 (a) A bitstring of ~~length~~ length 9 can have
$\underbrace{0 \; 1s}$ 1 1s 2 1s 3 1s, 4 1s, 5 1s ~~must~~

~~It a bit~~ must contain at least one occurrence of 00. (This is a pigeon hole principle.)

Now count

$\cancel{4}$ 1s : $\binom{5}{5}$ ways to place $\overset{5}{\cancel{\ }}$ 0s between 4 1s

5 1s : $\binom{6}{4}$ ways to place 4 0s between 5 1s

6 1s : $\binom{7}{3}$ ways to place 3 0s between 6 1s

7 1s : $\binom{8}{2}$ ways to place 2 0s between 7 1s

8 1s : $\binom{9}{1}$ ways to place 1 0 between 8 1s

7

(a) $\binom{5}{5}+\binom{6}{4}+\binom{7}{3}+\binom{8}{2}+\binom{9}{1}+\binom{10}{0}$ $\boxed{6}$

$= 1 + \frac{6\cdot 5}{2} + \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3} + \frac{8\cdot 7}{2} + 9 + 1$

$= 1 + 15 + 35 + 28 + 9 + 1$

$= 50 + 10 + 28 + 1 =$ **89** such bit strings

(b)

A is the set in (a)

B is the set of bit strings of length 9 that do not contain 11

$|A| = |B| = 89.$

$|A\cup B| = |A| + |B| - |A\cap B|$

$A\cap B = \{101010101, 010101010\}$

$|A\cup B| = 2 * 89 - 2 = \mathbf{176}$

$A\cup B$ the set of bit strings of length 9 that do not cont. 11 or 00

$(A\cup B)^c = A^c \cap B^c$ the set of bit strings of length 9 with at last one 00 and at least one 11

$|(A\cup B)^c| = 2^9 - 176 = 512 - 176 = 336$

8 a REARRANGE $\boxed{7}$

3 R 2 A 2 E

9 letters: $\dfrac{9!}{3!\,2!\,2!} =$

$$= \frac{9\cdot 8\cdot 7\cdot \cancel{6}\cdot 5\cdot \cancel{4}\cdot 3\cdot 2\cdot 1}{\cancel{3\cdot 2\cdot 1}\ \cancel{2\cdot 1}\ \cancel{2\cdot 1}}$$

$$= 9\cdot 56\cdot 30 = 270\cdot 56 = 15{,}120$$

b each permutation can be divided in two words in 8 ways. So

$15{,}120 * 8 = 120{,}960$

ways to write two words.

9 a two fances $\boxed{8}$

and 24 zeros spread inbetween.

So $\binom{26}{2} = \frac{26\cdot 25}{2} = 25*13$

$= 325$ solutions

b $\quad \underset{\text{place } 0}{\uparrow}\ 1\ \underset{\text{place } 0}{\uparrow}\ 1\ \underset{\text{place } 0}{\uparrow}$

remains to place 21 zero

So $\binom{23}{2}$ solutions

$= \frac{23\cdot 22}{2} = 23*11$

$\boxed{253}$

c All solutions

699 ✓ 897 ✓
789 ✓ 969 ✓
798 ✓ 978 ✓
879 ✓ 987 ✓
888 ✓ 996 ✓

only 10

Combinatorialy this is harder!

9

3 c A_1 $x_1 \geq 10$

10 0_s 1 1

put 14 remaining

$$|A_1| = \binom{16}{2} = 120$$

$$|A_1 \cup A_2 \cup A_3| = 3 * |A_1| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + \underbrace{|A_1 \cap A_2 \cap A_3|}_{=0}$$

$|A_1 \cap A_2|$

10 0_s 1 10 0_s 1

place 4 so $\binom{6}{2} = 15$

$$\underbrace{|A_1 \cup A_2 \cup A_3|}_{\text{at least one}} = 3 * 120 - 3 * 15 = \underline{\underline{315}}$$

$x_1 \geq 10$ or $x_2 \geq 10$ $x_3 \geq 10$

$(A_1 \cup A_2 \cup A_3)^c$ $x_1 \leq 9, x_2 \leq 9, x_3 \leq 9.$

$$|(A_1 \cup A_2 \cup A_3)^c| = 325 - 315 = 10.$$

10 ~~Letters~~ sent:
Emails $28 * 14 = 392$

How many pairs of students:

$$\binom{28}{2} = \frac{28 \cdot 27}{2} = 14 * 27 = \boxed{378}$$

put an email into the pair {a, b} if a wrote it and b received it or b wrote it and a received it

pigeons

pigeon holes

two emails belong to at least one pair of students, say {a, b}
This means that a wrote to b and b wrote to a.