10

Calculating the binomial coefficients we find out that if k is an odd number then the coefficient is 0. If k is even number between -50 and 50, then the coefficient is Binomial[100, (100-k)/2]. Test

$$\{\text{Coefficient}[\text{Expand}[(x + \frac{1}{x})^{100}], x^4], \text{Binomial}[100, (100 - 4) / 2]\}$$

{93206558875049876949581681100, 93206558875049876949581681100}

$$\{\text{Coefficient}[\text{Expand}[(x + \frac{1}{x})^{100}], x^{18}], \text{Binomial}[100, (100 - 18) / 2]\}$$

{20116440213369968050635175200, 20116440213369968050635175200}

To verify all the coefficients I use

Expand$\left[\left(x+\frac{1}{x}\right)^{100}\right]$

$$100891344545564193334812497256+\frac{1}{x^{100}}+\frac{100}{x^{98}}+\frac{4950}{x^{96}}+\frac{161700}{x^{94}}+\frac{3921225}{x^{92}}+$$
$$\frac{75287520}{x^{90}}+\frac{1192052400}{x^{88}}+\frac{16007560800}{x^{86}}+\frac{186087894300}{x^{84}}+\frac{1902231808400}{x^{82}}+$$
$$\frac{17310309456440}{x^{80}}+\frac{141629804643600}{x^{78}}+\frac{1050421051106700}{x^{76}}+\frac{7110542499799200}{x^{74}}+$$
$$\frac{44186942677323600}{x^{72}}+\frac{253338471349988640}{x^{70}}+\frac{1345860629046814650}{x^{68}}+$$
$$\frac{6650134872937201800}{x^{66}}+\frac{30664510802988208300}{x^{64}}+\frac{132341572939212267400}{x^{62}}+$$
$$\frac{535983370403809682970}{x^{60}}+\frac{2041841411062132125600}{x^{58}}+\frac{7332066885177656269200}{x^{56}}+$$
$$\frac{24865270306254660391200}{x^{54}}+\frac{79776075565900368755100}{x^{52}}+\frac{242519269720337121015504}{x^{50}}+$$
$$\frac{699574816500972464467800}{x^{48}}+\frac{1917353200780443050763600}{x^{46}}+\frac{4998813702034726525205100}{x^{44}}+$$
$$\frac{12410847811948286545336800}{x^{42}}+\frac{29372339821610944823963760}{x^{40}}+$$
$$\frac{66324638306863423796047200}{x^{38}}+\frac{143012501349174257560226775}{x^{36}}+$$
$$\frac{294692427022540894366527900}{x^{34}}+\frac{580717429720889409486981450}{x^{32}}+$$
$$\frac{1095067153187962886461165020}{x^{30}}+\frac{1977204582144932989443770175}{x^{28}}+$$
$$\frac{3420029547493938143902737600}{x^{26}}+\frac{5670048986634686922786117600}{x^{24}}+$$
$$\frac{9013924030034630492634340800}{x^{22}}+\frac{13746234145802811501267369720}{x^{20}}+$$
$$\frac{20116440213369968050635175200}{x^{18}}+\frac{28258808871162574166368460400}{x^{16}}+$$
$$\frac{38116532895986727945334202400}{x^{14}}+\frac{49378235797073715747364762200}{x^{12}}+$$
$$\frac{61448471214136179596720592960}{x^{10}}+\frac{73470998190814997343905056800}{x^{8}}+$$
$$\frac{84413487283064039501507937600}{x^{6}}+\frac{93206558875049876949581681100}{x^{4}}+$$
$$\frac{98913082887808032681188722800}{x^{2}}+98913082887808032681188722800\,x^{2}+$$
$$93206558875049876949581681100\,x^{4}+84413487283064039501507937600\,x^{6}+$$
$$73470998190814997343905056800\,x^{8}+61448471214136179596720592960\,x^{10}+$$
$$49378235797073715747364762200\,x^{12}+38116532895986727945334202400\,x^{14}+$$
$$28258808871162574166368460400\,x^{16}+20116440213369968050635175200\,x^{18}+$$
$$13746234145802811501267369720\,x^{20}+9013924030034630492634340800\,x^{22}+$$
$$5670048986634686922786117600\,x^{24}+3420029547493938143902737600\,x^{26}+$$
$$1977204582144932989443770175\,x^{28}+1095067153187962886461165020\,x^{30}+$$
$$580717429720889409486981450\,x^{32}+294692427022540894366527900\,x^{34}+$$
$$143012501349174257560226775\,x^{36}+66324638306863423796047200\,x^{38}+$$
$$29372339821610944823963760\,x^{40}+12410847811948286545336800\,x^{42}+$$
$$4998813702034726525205100\,x^{44}+1917353200780443050763600\,x^{46}+$$
$$699574816500972464467800\,x^{48}+242519269720337121015504\,x^{50}+79776075565900368755100\,x^{52}+$$
$$24865270306254660391200\,x^{54}+7332066885177656269200\,x^{56}+2041841411062132125600\,x^{58}+$$
$$535983370403809682970\,x^{60}+132341572939212267400\,x^{62}+30664510802988208300\,x^{64}+$$
$$6650134872937201800\,x^{66}+1345860629046814650\,x^{68}+253338471349988640\,x^{70}+$$
$$44186942677323600\,x^{72}+7110542499799200\,x^{74}+1050421051106700\,x^{76}+141629804643600\,x^{78}+$$
$$17310309456440\,x^{80}+1902231808400\,x^{82}+186087894300\,x^{84}+16007560800\,x^{86}+$$
$$1192052400\,x^{88}+75287520\,x^{90}+3921225\,x^{92}+161700\,x^{94}+4950\,x^{96}+100\,x^{98}+x^{100}$$

However, to get coefficients I have to turn the expression into a polynomial. Here is a smaller example

```
Expand[x^10 (x + 1/x)^10]
```

$1 + 10\,x^{2} + 45\,x^{4} + 120\,x^{6} + 210\,x^{8} + 252\,x^{10} + 210\,x^{12} + 120\,x^{14} + 45\,x^{16} + 10\,x^{18} + x^{20}$

```
CoefficientList[x^10 Expand[(x + 1/x)^10], x]
```

```
{1, 0, 10, 0, 45, 0, 120, 0, 210, 0, 252, 0, 210, 0, 120, 0, 45, 0, 10, 0, 1}
```

```
Table[{CoefficientList[Expand[x^100 (x + 1/x)^100], x][[2 (j + 50 + 1) - 1]], Binomial[100, 50 - j]},
  {j, -50, 50}]
```

```
{{1, 1}, {100, 100}, {4950, 4950}, {161700, 161700}, {3921225, 3921225},
 {75287520, 75287520}, {1192052400, 1192052400}, {16007560800, 16007560800},
 {186087894300, 186087894300}, {1902231808400, 1902231808400},
 {17310309456440, 17310309456440}, {141629804643600, 141629804643600},
 {1050421051106700, 1050421051106700}, {7110542499799200, 7110542499799200},
 {44186942677323600, 44186942677323600}, {253338471349988640, 253338471349988640},
 {1345860629046814650, 1345860629046814650}, {6650134872937201800, 6650134872937201800},
 {30664510802988208300, 30664510802988208300},
 {132341572939212267400, 132341572939212267400},
 {535983370403809682970, 535983370403809682970},
 {2041841411062132125600, 2041841411062132125600},
 {7332066885177656269200, 7332066885177656269200},
 {24865270306254660391200, 24865270306254660391200},
 {79776075565900368755100, 79776075565900368755100},
 {242519269720337121015504, 242519269720337121015504},
 {699574816500972464467800, 699574816500972464467800},
 {1917353200780443050763600, 1917353200780443050763600},
 {4998813702034726525205100, 4998813702034726525205100},
 {12410847811948286545336800, 12410847811948286545336800},
 {29372339821610944823963760, 29372339821610944823963760},
 {66324638306863423796047200, 66324638306863423796047200},
 {143012501349174257560226775, 143012501349174257560226775},
 {294692427022540894366527900, 294692427022540894366527900},
 {580717429720889409486981450, 580717429720889409486981450},
 {1095067153187962886461165020, 1095067153187962886461165020},
 {1977204582144932989443770175, 1977204582144932989443770175},
 {3420029547493938143902737600, 3420029547493938143902737600},
 {5670048986634686922786117600, 5670048986634686922786117600},
 {9013924030034630492634340800, 9013924030034630492634340800},
 {13746234145802811501267369720, 13746234145802811501267369720},
 {20116440213369968050635175200, 20116440213369968050635175200},
 {28258808871162574166368460400, 28258808871162574166368460400},
 {38116532895986727945334202400, 38116532895986727945334202400},
 {49378235797073715747364762200, 49378235797073715747364762200},
 {61448471214136179596720592960, 61448471214136179596720592960},
 {73470998190814997343905056800, 73470998190814997343905056800},
 {84413487283064039501507937600, 84413487283064039501507937600},
 {93206558875049876949581681100, 93206558875049876949581681100},
```

```
{98913082887808032681188722800, 98913082887808032681188722800},
{100891344545564193334812497256, 100891344545564193334812497256},
{98913082887808032681188722800, 98913082887808032681188722800},
{93206558875049876949581681100, 93206558875049876949581681100},
{84413487283064039501507937600, 84413487283064039501507937600},
{73470998190814997343905056800, 73470998190814997343905056800},
{61448471214136179596720592960, 61448471214136179596720592960},
{49378235797073715747364762200, 49378235797073715747364762200},
{38116532895986727945334202400, 38116532895986727945334202400},
{28258808871162574166368460400, 28258808871162574166368460400},
{20116440213369968050635175200, 20116440213369968050635175200},
{13746234145802811501267369720, 13746234145802811501267369720},
{9013924030034630492634340800, 9013924030034630492634340800},
{5670048986634686922786117600, 5670048986634686922786117600},
{3420029547493938143902737600, 3420029547493938143902737600},
{1977204582144932989443770175, 1977204582144932989443770175},
{1095067153187962886461165020, 1095067153187962886461165020},
{580717429720889409486981450, 580717429720889409486981450},
{294692427022540894366527900, 294692427022540894366527900},
{143012501349174257560226775, 143012501349174257560226775},
{66324638306863423796047200, 66324638306863423796047200},
{29372339821610944823963760, 29372339821610944823963760},
{12410847811948286545336800, 12410847811948286545336800},
{4998813702034726525205100, 4998813702034726525205100},
{1917353200780443050763600, 1917353200780443050763600},
{699574816500972464467800, 699574816500972464467800},
{242519269720337121015504, 242519269720337121015504},
{79776075565900368755100, 79776075565900368755100},
{24865270306254660391200, 24865270306254660391200},
{7332066885177656269200, 7332066885177656269200},
{2041841411062132125600, 2041841411062132125600},
{535983370403809682970, 535983370403809682970},
{132341572939212267400, 132341572939212267400},
{30664510802988208300, 30664510802988208300},
{6650134872937201800, 6650134872937201800}, {1345860629046814650, 1345860629046814650},
{253338471349988640, 253338471349988640}, {44186942677323600, 44186942677323600},
{7110542499799200, 7110542499799200}, {1050421051106700, 1050421051106700},
{141629804643600, 141629804643600}, {17310309456440, 17310309456440},
{1902231808400, 1902231808400}, {186087894300, 186087894300},
{16007560800, 16007560800}, {1192052400, 1192052400}, {75287520, 75287520},
{3921225, 3921225}, {161700, 161700}, {4950, 4950}, {100, 100}, {1, 1}}
```

Or, even better,

```
Table[CoefficientList[Expand[x^100 (x + 1/x)^100], x][[2 (j + 50 + 1) - 1]] - Binomial[100, 50 - j],
  {j, -50, 50}]
```

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

14

First prove that Binomial[n,Floor[n/2]] = Binomial[n,Ceiling[n/2]]. First check the first few terms.

```
Table[{Binomial[n, Floor[n / 2]], Binomial[n, Ceiling[n / 2]]}, {n, 1, 20}]

{{1, 1}, {2, 2}, {3, 3}, {6, 6}, {10, 10}, {20, 20}, {35, 35}, {70, 70}, {126, 126},
 {252, 252}, {462, 462}, {924, 924}, {1716, 1716}, {3432, 3432}, {6435, 6435},
 {12870, 12870}, {24310, 24310}, {48620, 48620}, {92378, 92378}, {184756, 184756}}
```

If n is even, then $n = 2k$ for some positive integer k. In this case Floor[n] = Ceiling[n] = k.

If n is odd, then $n = 2k - 1$ for some positive integer k. In this case Floor[n] = k-1 and Ceiling[n] = k. Since for every j Binomial[n,j] = Binomial[n, n-j], we have Binomial[n,k-1] = Binomial[n, n-k+1] = Binomial[n, 2 k-1 -k+1] = Binomial[n, k].

Next, let 1≤ k < Floor[n/2]. Then 2k < 2 Floor[n/2]. Consequently, 2k+1 ≤ 2 Floor[n/2]. Since these two numbers can not be equal (one is even, the other odd), we have 2k+1 < 2 Floor[n/2]. Clearly 2 Floor[n/2] ≤ n, and thus 2 k + 1 < n. This yields, k+1 < n-k, and hence

$\frac{1}{k+1} > \frac{1}{n-k}$.

Multiplying both sides by $\frac{n!}{k!\,(n-k-1)!}$ we get $\frac{n!}{(k+1)!\,(n-k-1)!} > \frac{n!}{k!\,(n-k)!}$, that is Binomial[n,k+1] > Binomial[n,k].

15

The number Binomial[n,k] represents the number of bit strings of length n with exactly k zeros. The number 2^n represents the number of all bit strings of length n. Thus the inequality.

16

```
Table[{N[Binomial[n, Floor[n / 2]], 2], N[2^n / (n), 2]}, {n, 2, 100}]
```

{{2.0, 2.0}, {3.0, 2.7}, {6.0, 4.0}, {10., 6.4}, {20., 11.}, {35., 18.}, {70., 32.},
{1.3×10^{2}, 57.}, {2.5×10^{2}, 1.0×10^{2}}, {4.6×10^{2}, 1.9×10^{2}}, {9.2×10^{2}, 3.4×10^{2}},
{1.7×10^{3}, 6.3×10^{2}}, {3.4×10^{3}, 1.2×10^{3}}, {6.4×10^{3}, 2.2×10^{3}}, {1.3×10^{4}, 4.1×10^{3}},
{2.4×10^{4}, 7.7×10^{3}}, {4.9×10^{4}, 1.5×10^{4}}, {9.2×10^{4}, 2.8×10^{4}}, {1.8×10^{5}, 5.2×10^{4}},
{3.5×10^{5}, 1.0×10^{5}}, {7.1×10^{5}, 1.9×10^{5}}, {1.4×10^{6}, 3.6×10^{5}}, {2.7×10^{6}, 7.0×10^{5}},
{5.2×10^{6}, 1.3×10^{6}}, {1.0×10^{7}, 2.6×10^{6}}, {2.0×10^{7}, 5.0×10^{6}}, {4.0×10^{7}, 9.6×10^{6}},
{7.8×10^{7}, 1.9×10^{7}}, {1.6×10^{8}, 3.6×10^{7}}, {3.0×10^{8}, 6.9×10^{7}}, {6.0×10^{8}, 1.3×10^{8}},
{1.2×10^{9}, 2.6×10^{8}}, {2.3×10^{9}, 5.1×10^{8}}, {4.5×10^{9}, 9.8×10^{8}}, {9.1×10^{9}, 1.9×10^{9}},
{1.8×10^{10}, 3.7×10^{9}}, {3.5×10^{10}, 7.2×10^{9}}, {6.9×10^{10}, 1.4×10^{10}}, {1.4×10^{11}, 2.7×10^{10}},
{2.7×10^{11}, 5.4×10^{10}}, {5.4×10^{11}, 1.0×10^{11}}, {1.1×10^{12}, 2.0×10^{11}}, {2.1×10^{12}, 4.0×10^{11}},
{4.1×10^{12}, 7.8×10^{11}}, {8.2×10^{12}, 1.5×10^{12}}, {1.6×10^{13}, 3.0×10^{12}}, {3.2×10^{13}, 5.9×10^{12}},
{6.3×10^{13}, 1.1×10^{13}}, {1.3×10^{14}, 2.3×10^{13}}, {2.5×10^{14}, 4.4×10^{13}}, {5.0×10^{14}, 8.7×10^{13}},
{9.7×10^{14}, 1.7×10^{14}}, {1.9×10^{15}, 3.3×10^{14}}, {3.8×10^{15}, 6.6×10^{14}}, {7.6×10^{15}, 1.3×10^{15}},
{1.5×10^{16}, 2.5×10^{15}}, {3.0×10^{16}, 5.0×10^{15}}, {5.9×10^{16}, 9.8×10^{15}}, {1.2×10^{17}, 1.9×10^{16}},
{2.3×10^{17}, 3.8×10^{16}}, {4.7×10^{17}, 7.4×10^{16}}, {9.2×10^{17}, 1.5×10^{17}}, {1.8×10^{18}, 2.9×10^{17}},
{3.6×10^{18}, 5.7×10^{17}}, {7.2×10^{18}, 1.1×10^{18}}, {1.4×10^{19}, 2.2×10^{18}}, {2.8×10^{19}, 4.3×10^{18}},
{5.6×10^{19}, 8.6×10^{18}}, {1.1×10^{20}, 1.7×10^{19}}, {2.2×10^{20}, 3.3×10^{19}}, {4.4×10^{20}, 6.6×10^{19}},
{8.7×10^{20}, 1.3×10^{20}}, {1.7×10^{21}, 2.6×10^{20}}, {3.4×10^{21}, 5.0×10^{20}}, {6.9×10^{21}, 9.9×10^{20}},
{1.4×10^{22}, 2.0×10^{21}}, {2.7×10^{22}, 3.9×10^{21}}, {5.4×10^{22}, 7.7×10^{21}}, {1.1×10^{23}, 1.5×10^{22}},
{2.1×10^{23}, 3.0×10^{22}}, {4.2×10^{23}, 5.9×10^{22}}, {8.4×10^{23}, 1.2×10^{23}}, {1.7×10^{24}, 2.3×10^{23}},
{3.3×10^{24}, 4.6×10^{23}}, {6.6×10^{24}, 9.0×10^{23}}, {1.3×10^{25}, 1.8×10^{24}}, {2.6×10^{25}, 3.5×10^{24}},
{5.2×10^{25}, 7.0×10^{24}}, {1.0×10^{26}, 1.4×10^{25}}, {2.1×10^{26}, 2.7×10^{25}}, {4.1×10^{26}, 5.4×10^{25}},
{8.1×10^{26}, 1.1×10^{26}}, {1.6×10^{27}, 2.1×10^{26}}, {3.2×10^{27}, 4.2×10^{26}}, {6.4×10^{27}, 8.3×10^{26}},
{1.3×10^{28}, 1.6×10^{27}}, {2.5×10^{28}, 3.2×10^{27}}, {5.0×10^{28}, 6.4×10^{27}}, {1.0×10^{29}, 1.3×10^{28}}}

First notice that for $n \geq 2$ we have Binomial[n,Floor[n/2]] ≥ Binomial[n,0] + Binomial[n,1]. Therefore

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{n} + \sum_{k=1}^{n-1} \binom{n}{k} \leq \binom{n}{\text{Floor}[n/2]} + (n-1)\binom{n}{\text{Floor}[n/2]} = n\binom{n}{\text{Floor}[n/2]}$$

22

This is counting number of committees with a subcommittees. For example the university faculty elects the faculty senate, then the senate elects its executive committee. If there are n faculty members, the senate has r members and the executive committee of the senate has k members count number of different ways of forming the senate with its executive committee.

One way of counting: Choose the senate, then choose the executive committee from the senate. By the product rule there are Binomial[n,r]*Binomial[r,k] ways to do this.

Another way of counting is: choose the executive committee from the whole faculty, then choose the remaining members of the senate. By the product rule there are Binomial[n,k]*Binomial[n-k,r-k] ways to do this.

Since we count the same set these two numbers must be equal.

25

A combinatorial way to prove this identity is to look at a class of 2n males and 2 females and count the number of committees of n+1 members that can be formed. There are Binomial[2n+2,n+1] such committees.

Now we count the number of committees based of number of female members.

There are Binomial[2n,n+1] committees with no female members, there are 2 Binomial[2n,n] with one female member and thee are Binomial[2n,n-1] committees with two female members. Since Binomial[2n,n-1] = Binomial[2n,n+1], the formula follows.

26

The formula

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \sum_{k=1}^{n} \binom{n}{k}\binom{n}{n+1-k} = \binom{2n}{n+1}$$

since both sides count the number of committees with n+1 memberf from a class of 2n people. In the sum we assume that there are n males and n females and then count the committees with 1, 2, 3, ..., n males.

Now we need to prove

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1} \Big/ 2$$

or, simpler,

$$\binom{2n+2}{n+1} = 2\binom{2n}{n+1} + 2\binom{2n}{n}$$

This can be proved by using Pascal's identity twice

$$\binom{2n+2}{n+1} = \binom{2n+1}{n+1} + \binom{2n+1}{n} =$$

$$\binom{2n}{n+1} + \binom{2n}{n} + \binom{2n}{n} + \binom{2n}{n-1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{2n-(n-1)} = 2\binom{2n}{n+1} + 2\binom{2n}{n}$$