MATH 209

Examination 1
April 19, 2012

Name Key

1. Let p and q be propositions.

   (a) Prove that the negation of $p \oplus q$ is $p \leftrightarrow q$.

   (b) Consider the proposition $(p \to q) \oplus (q \to p)$. Find an equivalent, but much simpler proposition.

2. (a) State the definition of a rational number. State the definition of an irrational number.

   (b) Prove or disprove the following theorem: If a is rational and b is irrational, then ab is irrational.

3. The universe of discourse in this problem is the set of all real numbers.

   Consider the following proposition:

   "For every x there exists y such that for all z we have $z < y$ implies $z^2 > x^2$."

   (a) Write the given proposition using quantifiers.

   (b) State the negation of the proposition in (3a).

   (c) Decide which proposition is true: the proposition in (3a) or the proposition in (3b). Prove your claim.

4. Let A and B be sets. The set $A \oplus B$ is defined as $A \oplus B = \{x \mid (x \in A) \oplus (x \in B)\}$.

   (a) Let A, B, C be given sets. Use a Venn diagram to represent the set $(A \oplus B) \oplus C$.

   (b) Find a formula for the set represented by the Venn diagram in Figure 1. For the full credit you must use the set in (4a).

5. What is wrong with the proof given in the box below? Please be specific.

$$\frac{25}{36} = \frac{9+16}{36} \qquad (1)$$

$$\frac{25}{36} = \frac{1}{4} + \frac{4}{9} \qquad (2)$$

$$\frac{1}{36} = \frac{1}{4} - 2\frac{1}{2}\frac{2}{3} + \frac{4}{9} \qquad (3)$$

$$\left(\frac{1}{6}\right)^2 = \left(\frac{1}{2} - \frac{2}{3}\right)^2 \qquad (4)$$

$$\frac{1}{6} = \frac{1}{2} - \frac{2}{3} \qquad (5)$$

$$1 = 3 - 4 \qquad (6)$$

$$1 = -1 \qquad (7)$$

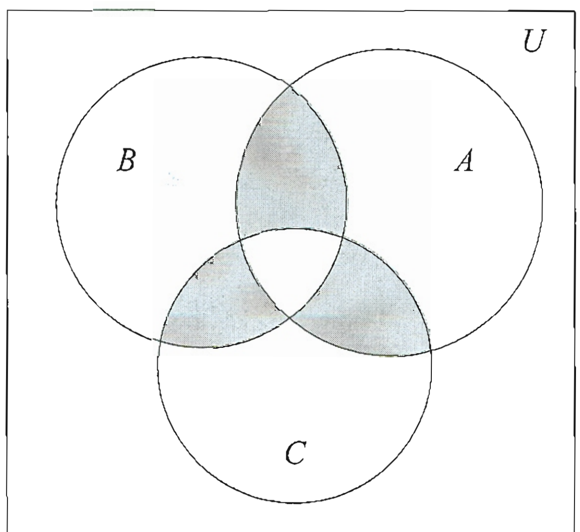


Figure 1: Venn diagram for (4b)

1 a Here is the truth table that $\boxed{1}$ proves the claim

p	q	$p \oplus q$	$\neg p \oplus q$	$p \leftrightarrow q$
F	F	F	T	T
F	T	T	F	F
T	F	T	F	F
T	T	F	T	T

identical truth tables

b

p	q	$(p \to q) \oplus (q \to p)$	$p \oplus q$
F	F	F	F
F	T	T	T
T	F	T	T
T	T	F	F

equivalent

2 a A real number a is rational iff there exist integers p and q such that $q \neq 0$ and $a = p/q$.

A real number is irrational if it is Not rational.

2 b This is not true.

$a = 0$ is rational

$b = \sqrt{2}$ is irrational

$a \cdot b = 0$ is rational.

Thus a is rational and b is irrational and $a \cdot b$ is rational is possible.

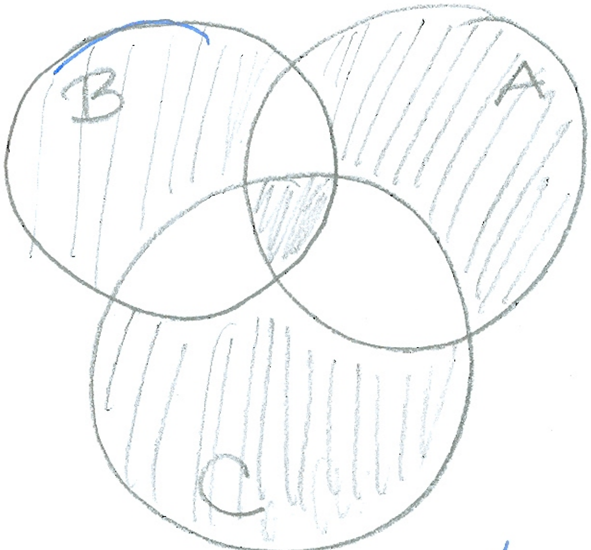
3 a $\forall x \, \exists y \, \forall z \; (z < y) \rightarrow (z^2 > x^2)$

b $\exists x \, \forall y \, \exists z \; (z < y) \wedge (z^2 \leq x^2)$

c a is true. For $x \in \mathbb{R}$ set $y = -|x|$. then

$$z < -|x| \rightarrow -z > |x| \rightarrow$$
$$\rightarrow z^2 > |x|^2$$
$$\rightarrow z^2 > x^2.$$

4 a



b The given set is
$$(A \cup B \cup C) - (A \oplus B \oplus C).$$

5 The line (4) does not imply the line (5). The implication
$$a^2 = b^2 \Rightarrow a = b$$
is Not true. In this case
$$\underbrace{\left(\frac{1}{6}\right)^2 = \left(-\frac{1}{6}\right)^2}_{T} \rightarrow \underbrace{\frac{1}{6} = -\frac{1}{6}}_{F}$$