MATH 224 Examination 1
October 14, 2009

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
There are four problems. Each is worth 25 points.

1. The graph on the right shows the contour line at level 1 of a plane. This plane also passes through the point $(1,1,2)$.

(a) Add and label the contour line of this plane at the level 2.

(b) Calculate slope m in the x-direction and slope n in the y-direction of this plane.

(c) Find the equation, $z = mx + ny + c$, of this plane.

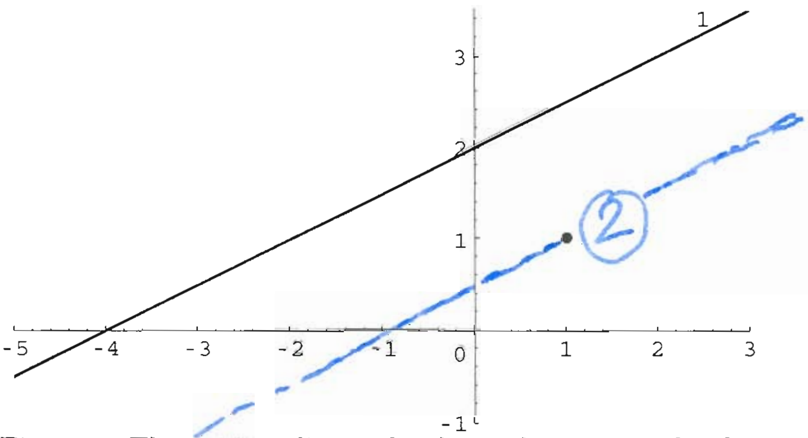


Figure 1: The contour line at level 1 and a point at level 2

2. The graph on the right shows three points $A = (2,0,0)$, $B = (0,3,0)$ and $C = (0,0,1)$.

(a) Resolve the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$ into components using the vectors $\overrightarrow{i}$, $\overrightarrow{j}$, $\overrightarrow{k}$.

(b) Use the cross product to find a vector orthogonal to the plane drawn through the points A, B and C.

(c) Calculate the area of the triangle ABC. (Notice that you already this area is closely related to the magnitude of the vector calculated in (2b).)

The items (2b) and (2c) can be calculated without using the cross product. You will get the full credit if you do it correctly and explain your method.

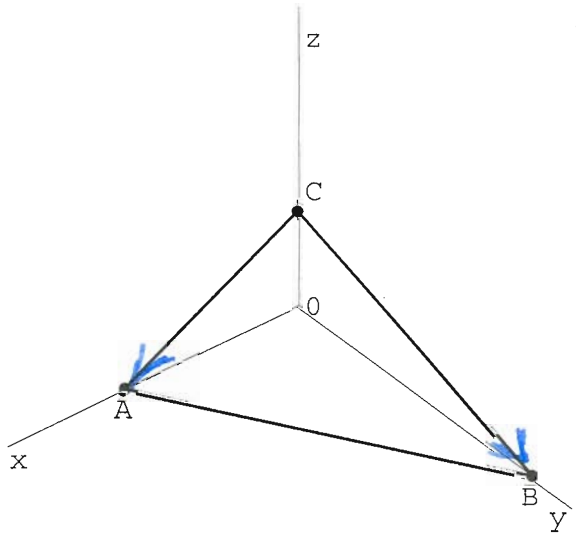


Figure 2: Three points and the plane

3. Consider the vectors $\overrightarrow{u} = -\overrightarrow{i} + 3\overrightarrow{j} + 2\overrightarrow{k}$ and $\overrightarrow{v} = 5\overrightarrow{i} - 5\overrightarrow{j} - 4\overrightarrow{k}$.

(a) Calculate $\|\overrightarrow{u}\|$ and $\|\overrightarrow{v}\|$. Calculate $\overrightarrow{u} \cdot \overrightarrow{v}$.

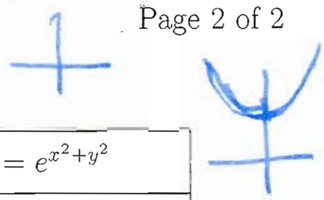
(b) Denote by θ the angle between vectors $\overrightarrow{u}$ and $\overrightarrow{v}$. Calculate $\cos\theta$. Give both exact and approximate value for this quantity.

Determine which of the angles $0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{3\pi}{4}, \frac{7\pi}{8}, \pi$ is the closest to the angle θ.

(c) Write the vector $\overrightarrow{v}$ as the sum of two vectors, one parallel and one perpendicular to $\overrightarrow{u}$.

OVER

e^{-x^2}

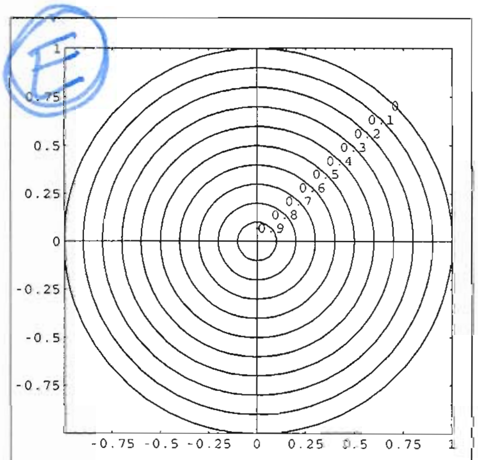


4. I used six of the following nine functions

~~(A)~~ $z = \ln(\sqrt{x^2+y^2})$	(B) ✓ $z = \sqrt{x^2+y^2}$	~~(C)~~ $z = e^{x^2+y^2}$
(D) ✓ $z = e^{-x^2-y^2}$	(E) ✓ $z = 1 - \sqrt{x^2+y^2}$	(F) $z = -\ln(\sqrt{x^2+y^2})$
(G) ✓ $z = x^2+y^2$	~~(H)~~ $z = \sqrt{x^2+y^2} - 1$	(I) ✓ $z = \sqrt{1-x^2-y^2}$

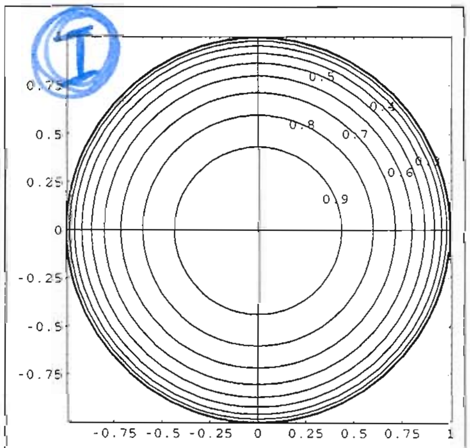
to plot the contour diagrams below. The contour diagrams below show only the contour lines at levels $0, 0.1, 0.2, \ldots, 0.9$ and 1. If a contour line consists of only one point it is not represented in the figure.

Identify which contour diagram belongs to which function. Place the letter corresponding to the function in the box with the contour diagram of that function.



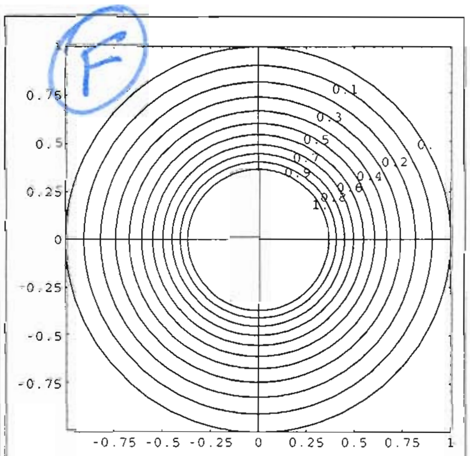
E

Figure 3



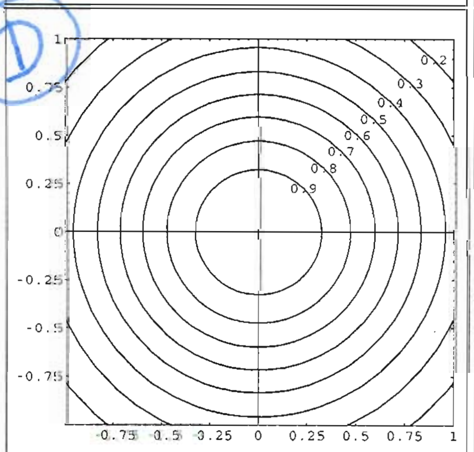
I

Figure 4



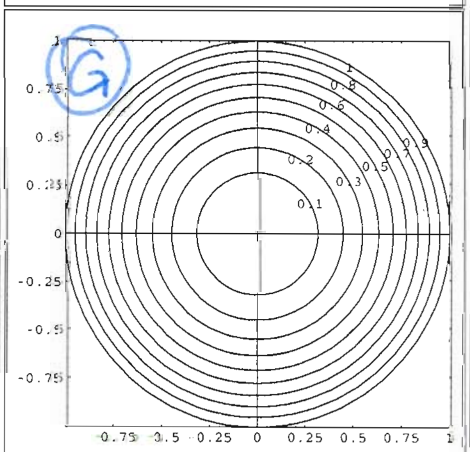
F

Figure 5



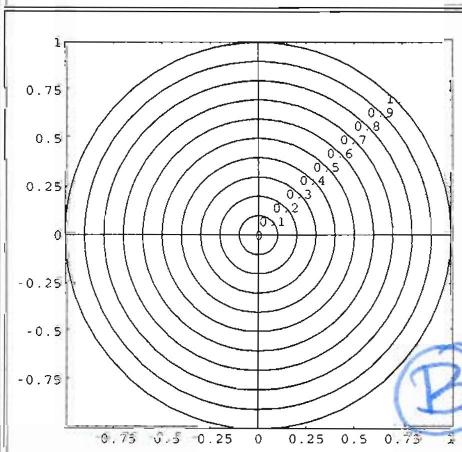
D

Figure 6



G

Figure 7



B

Figure 8

1 b The equation of the given contour line is

$$y = \frac{1}{2}x + 2$$

Two specific points on this line are $(1, 5/2)$ and $(-2, 1)$

So the points $(1, 5/2, 1)$ and $(-2, 1, 1)$ are on the plane.

Hence, m is calculated from $(1, 1, 2)$ and $(-2, 1, 1)$

$$m = \frac{2-1}{1-(-2)} = \frac{1}{3}$$

and n is calculated from $(1, 1, 2)$ and $(1, 5/2, 1)$

$$n = \frac{2-1}{1-5/2} = \frac{1}{-3/2} = -\frac{2}{3}$$

$$z = \frac{1}{3}(x-1) - \frac{2}{3}(y-1) + 2$$

$$= \frac{1}{3}x - \frac{2}{3}y - \frac{1}{3} + \frac{2}{3} + 2 = \boxed{\frac{1}{3}x - \frac{2}{3}y + \frac{7}{3}}$$

$\boxed{1}$

② ⓐ $\overrightarrow{CA} = 2\vec{i} + 0\vec{j} - \vec{k}$ $\boxed{2}$

$\overrightarrow{CB} = 0\vec{i} + 3\vec{j} - \vec{k}$

ⓑ $\overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} =$

$= 3\vec{i} - \vec{j}\left(2(-1) - 0\right) + \vec{k}\,6$

$= 3\vec{i} + 2\vec{j} + 6\vec{k}$

ⓒ The area is $\frac{1}{2}\sqrt{9+4+36} = \frac{7}{2}$

Alternative ⓑ: the equation of the plane through A, B, C: $m = -1/2$, $n = -1/3$

$z = -\frac{1}{2}x - \frac{1}{3}y + 1$ or

$3x + 2y + 6z = 1$, so $\vec{n} = 3\vec{i} + 2\vec{j} + 6\vec{k}$

Alternative ⓒ Calculate θ the angle between $\overrightarrow{CA}, \overrightarrow{CB}$:

$$\cos\theta = \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\|\overrightarrow{CA}\| \cdot \|\overrightarrow{CB}\|} = \frac{1}{\sqrt{5}\sqrt{10}} = \frac{1}{5\sqrt{2}}$$

$$\text{Area} = \frac{1}{2}\|\overrightarrow{CA}\| \cdot \|\overrightarrow{CB}\| \cdot \sin\theta = \frac{1}{2}\sqrt{5}\sqrt{10} \cdot \sqrt{1 - \frac{1}{2 \cdot 25}} =$$

$$= \frac{1}{2}\,5\sqrt{2}\,\sqrt{50-1}\,/\,5\sqrt{2} = 7/2$$

(3) (a) $\|\vec{u}\| = \sqrt{14}$, $\|\vec{v}\| = \sqrt{66}$ $\boxed{3}$

$$\vec{u} \cdot \vec{v} = -5 - 15 - 8 = -28$$

(b) $$\cos\theta = -\frac{28}{2\sqrt{7}\sqrt{33}} = -\frac{14}{\sqrt{7}\sqrt{33}} = -\frac{2\sqrt{7}}{\sqrt{33}}$$

$$\theta = \arccos\left(-2\sqrt{\frac{7}{33}}\right) \approx 2.7418$$

$$\frac{2.7418}{\pi} = 0.873 \text{ close to } 0.875 = 7/8$$

The closest angle is $7\pi/8$

(c) $\vec{v} = t\vec{u} + \vec{w}$ where $\vec{w} \perp \vec{u}$

$$\vec{u} \cdot \vec{v} = t\,\|\vec{u}\|^2 \qquad t = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}$$

$$t = -\frac{28}{14} = -2$$

$$\vec{v} = \boxed{-2(-\vec{i} + 3\vec{j} + 2\vec{k})} + \underbrace{\boxed{3\vec{i} + \vec{j}}}_{\perp \vec{u}}$$

$$\vec{v} = \underbrace{2\vec{i} - 6\vec{j} - 4\vec{k}}_{-2\vec{u}} + \underbrace{(3\vec{i} + \vec{j})}_{\perp \vec{u}}$$

$$(3\vec{i} + \vec{j}) \perp \vec{u}$$