MATH 224 Examination 2
November 4, 2009

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
There are four problems. Each is worth 25 points.

1. The monthly mortgage payment P (in dollars) for a 30-year loan is a function of two variables the loan amount L and the interest rate r: $P = f(L, r)$. In the rest of the problem we assume that L is given in thousands of dollars, r is given in percentages and P is in dollars.

   (a) Explain the financial significance of the numerical information given below:

   $$\text{(A)} \quad f(215, 6) = 1289.03; \qquad \text{(B)} \quad \left.\frac{\partial f}{\partial r}\right|_{(L,6)} = 0.64\,L; \qquad \text{(C)} \quad \left.\frac{\partial f}{\partial L}\right|_{(L,6)} = 6.$$

   For each number in (A), (B) and (C) above provide the corresponding units.

   (b) Find a local linearization of the function $f(L, r)$ near the point $(215, 6)$.

   (c) Assume that you plan to borrow between 200 and 230 thousand dollars. Assume also that the interest rate fluctuates between 5.5% and 6.5%. Give an estimate for the lowest and the highest monthly mortgage payment P under these assumptions and using the information given above.

2. Consider the function $f(x, y) = x^2y - 2\sqrt{y}$. (You can think of f as being a temperature at each point of a heated plate.) Consider the point $P = (2, 1)$.

   (a) Find the vector in the direction of maximum rate of change of f at P. What is the maximum rate of change of f?

   (b) Find the instantaneous rate of change of f as you leave P heading toward $(1, 4)$.

   (c) Find a vector in a direction in which the rate of change of f at P is 0.

   (d) Find two directions in which the rate of change of f at P is 4.

3. Consider the plane $2x - 3y - 6z = 21$ and the sphere $x^2 + y^2 + z^2 = 4$.

   (a) Calculate the distance from the origin to the given plane.

   (b) Based on the calculation in (3a) you can answer whether the given plane is a tangent plane to the given sphere. Explain.

   (c) Find a point on the sphere at which the tangent plane is parallel to the given plane.

   (d) Find the equation of the tangent plane from (3c).

4. Find all critical points of the function $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$. There are four of them. Classify the critical points as local minima, local maxima or saddle points.

① ⓐ Ⓐ If you borow 215K at 6% interest your monthly payment is \$1289.03. $\boxed{1}$

Ⓑ As rate change near 6% your montly payment will change at the rate 0.64L \$/% ; this means, if you borow L thousands of \$ at 6.1% your payment would increase by 0.1*0.64L \$

ⓒ If you borow more you will pay more \$6/\$1000. units are \$/1000\$

ⓑ

$$f(L,r) \approx 1289.03 + 0.64*215(r-6) + 6(L-215)$$

ⓒ Lowest : $1289.03 + 0.64*215(-0.5) + 6(-15)$

$\boxed{1130.23}$

highest : $1289.03 + 0.64*215(0.5) + 6*15$

$\boxed{1447.83}$

ⓑ $f(L,r) \approx 1289.03 + 137.6(r-6) + 6(L-215)$

$\approx 137.6r + 6L - 826.57$

(2)

$f_x = 2xy - 0$ $\quad P = (2,1)$ $\quad \boxed{2}$

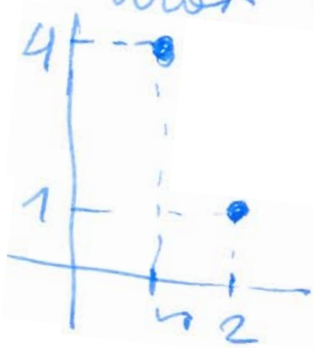
$f_y = x^2 - \frac{1}{\sqrt{y}}$ $\quad f_x(P) = 4$

$f_y(P) = 3$

(a) $\vec{\nabla} f(P) = 4\vec{i} + 3\vec{j}$

This is the vector in the direction of max change. The max change is 5.

(b)



$\vec{u} = (-\vec{i} + 3\vec{j})/\sqrt{10}$

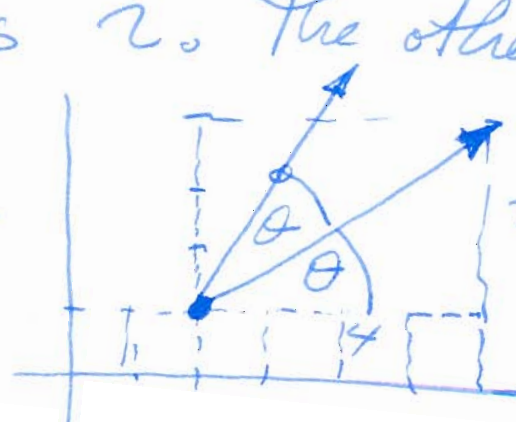
The change is

$$-\frac{4}{\sqrt{10}} + \frac{9}{\sqrt{10}} = \sqrt{\frac{5}{2}}$$

(c) It is the vector orthogonal to $4\vec{i} + 3\vec{j}$. That is

$3\vec{i} - 4\vec{j}$ OR $-3\vec{i} + 4\vec{j}$

(d) One direction is $\vec{i}$. The other direction is at the angle $\arctan \frac{3}{4}$ from $4\vec{i} + 3\vec{j}$ as pictured.



We are looking for the direction $\boxed{2_a}$

$\vec{i} + t\vec{j}$ such that $\frac{1}{\sqrt{1+t^2}}(\vec{i}+t\vec{j})(4\vec{i}+3\vec{j}) = 4$

So, solve for t:

$$4+3t = 4\sqrt{1+t^2} \;\Big/^{2}$$

$$16+24t+9t^2 = 16(1+t^2)$$

$$7t^2 - 24t = 0$$

$t=0$ and $t = 24/7$

So the directions are

$\vec{i}$ and $7\vec{i} + 24\vec{j}$

③ $\boxed{3}$

ⓐ $2x-3y-6z=21$

One point in the plane $P=(0,-1,-3)$

The unit normal vector is $\vec{n}=\frac{1}{7}(2\vec{i}-3\vec{j}-6\vec{k})$

The distance is

$$\vec{n}\cdot\vec{OP}=\frac{1}{7}(3+18)=\frac{21}{7}=3$$

ⓑ The sphere has radius 2, so it is not tangent to the plane. The plane is too far from the origin

$3>2$.

ⓒ The gradient to the sphere is

$$2x\vec{i}+2y\vec{j}+2z\vec{k}=\lambda(2\vec{i}-3\vec{j}-6\vec{k})$$

gradient $\parallel \vec{n}$

$2x=2\lambda$
$2y=-3\lambda$
$2z=-6\lambda$

$$\lambda^2+\frac{9}{4}\lambda^2+\frac{36}{4}\lambda^2=4$$

$$\boxed{\lambda=\pm\frac{4}{7}}$$

The points are $\left(\pm\frac{4}{7},\mp\frac{6}{7},\mp\frac{12}{7}\right)$

(d) The plane tangent to the sphere is $\boxed{4}$

$$2\left(x-\frac{4}{7}\right)-3\left(y+\frac{6}{7}\right)-6\left(z+\frac{12}{7}\right)=0$$

that is $2x-3y-6z-\underbrace{\frac{8+18+72}{7}}_{14}=0$

$$\boxed{2x-3y-6z=14}$$

(4) $f_x = 6xy-6x = 6x(y-1)$

$f_y = 3x^2+3y^2-6y$

$f_x=0$ yields $x=0$ OR $y=1$

$3y^2-6y=0$

$y=0$

OR $y=2$

$3x^2-3=0$

$x=1$

or $x=-1$

CRITICAL POINTS

$(0,0)$ $(0,2)$, $(-1,1)$, $(1,1)$

Max Min Saddle Saddle

$\boxed{5}$

	(0,0)	(0,2)	(-1,1)	(1,1)
f_{xx} $6y-6$	-1	6	0	0
f_{xy} $6x$	0	0	-6	6
f_{yy} $6y-6$	-1	6	0	0
D	>0	>0	<0	<0
	Max	Min	Saddle	Saddle

Back to Pr. 3 (d). Here is an easier way to determine the tangent plane. We know that the tangent plane has the form

$$2x - 3y - 6z = c$$

We will calculate c so that the distance to the origin will be 2. A point in this plane is $(c/2, 0, 0)$ and $\vec{n} = \frac{1}{7}(2\vec{i} - 3\vec{j} - 6\vec{k})$

So the distance is

$$2 = \frac{c}{2}\frac{2}{7},$$ so $c = 14$. Also $c = -14$ will work.

$$\boxed{2x - 3y - 6z = 14} \text{ or } -14$$