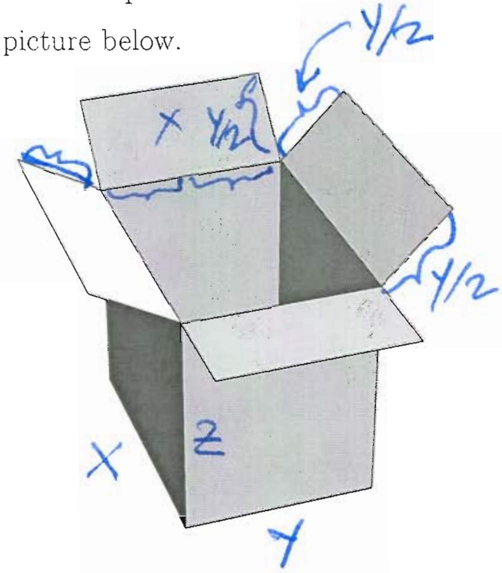
MATH 224

Examination 3
November 25, 2009

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
There are five problems. Each is worth 20 points.

1. A design of a standard moving box is given on the picture below.

   Use Multivariable Calculus to design a cardboard box with the volume of 1 cubic unit and with the minimum surface area. of the mat

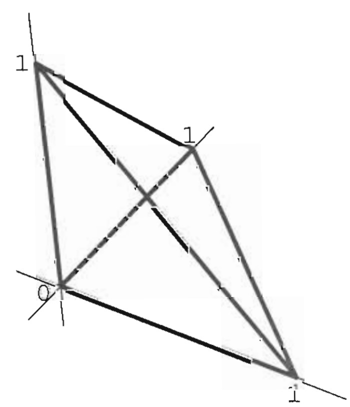
   Label the edges on the picture to the right with x, y, z.

   Calculate the exact values of x, y, z which give the minimum area.

   (Notice that the bottom of the box is constructed in the same way as the top. This is not shown in the picture.)

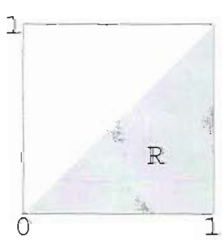
2. Find the minimum and maximum values of the function $f(x, y, z) = x^2 - y + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

3. The pyramid in the picture to the right is bounded by the coordinate planes $x = 0, y = 0, z = 0$ and the plane $x + y + z = 1$.

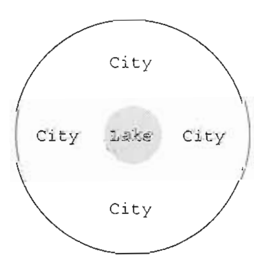
   Set up the triple integral to evaluate the volume of this pyramid.

   Evaluate this triple integral.



4. Evaluate $\int_R e^{-x^2} dA$, where R is given in the picture to the left. (It is essential to choose the order of integration correctly.)

5. The picture on the right represents a simplified city with a lake in its center. Everybody wants to live close to the lake. The radius of the lake is 1km and the radius of the city is 4km. Calculate the average distance from the lake in this city.



① Volume $xyz = 1$, $z = \frac{1}{xy}$ $\boxed{1}$

The surface area of the material

$$2(x+y)(z+y) = 2(x+y)z + 2(x+y)y$$

with $z = \frac{1}{xy}$

$$S(x,y) = \frac{2}{x} + \frac{2}{y} + 2xy + 2y^2$$

$$S_x = -\frac{2}{x^2} + 2y = 0, \quad \boxed{y = \frac{1}{x^2}}$$

$$S_y = -\frac{2}{y^2} + 2x + 4y = 0$$

Use $y = \frac{1}{x^2}$, $\quad -x^4 + 2x + \frac{2}{x^2} = 0$

$$\mp x^6 \pm x^3 \pm 2 = 0$$

$$x^3 = \frac{1 \pm \sqrt{1+8}}{2} = 2$$

$$x = \sqrt[3]{2}, \quad y = \frac{1}{\sqrt[3]{4}} = \frac{1}{2}\sqrt[3]{2} = \boxed{\frac{1}{2}x = y}$$

$$z = \frac{1}{xy} = \frac{2}{\sqrt[3]{2}\,\sqrt[3]{2}} = \boxed{\sqrt[3]{2} = x} = z$$

2 $\qquad x^2+y^2+z^2=1 \qquad \boxed{2}$

$$\cancel{2}x = \lambda \cancel{2}x$$

$$-1 = \lambda 2y \Rightarrow y = -\frac{1}{2\lambda}$$

$$\boxed{z=-y} \qquad 1 = \lambda 2z \Rightarrow z = \frac{1}{2\lambda}$$

$$x(1-\lambda)=0$$

$x=0$ OR $\lambda = 1$

Case $x=0$:

$$2y^2 = 1$$

$$y^2 = \frac{1}{2}$$

$$y = \frac{1}{\sqrt{2}}, \ y = -\frac{1}{\sqrt{2}}$$

$$z = -\frac{1}{\sqrt{2}}, \ z = \frac{1}{\sqrt{2}}$$

$$\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \ P1$$

$$\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \ P2$$

Case $\lambda = 1$:

$$y = -\frac{1}{2}, \ z = \frac{1}{2}$$

$$x^2 + \frac{1}{4} + \frac{1}{4} = 1$$

$$x = \frac{1}{\sqrt{2}}, \ x = -\frac{1}{\sqrt{2}}$$

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right) \ P3$$

$$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right) \ P4$$

Values: $f(P1) = -\sqrt{2}$, $f(P2) = \sqrt{2}$

$f(P3) = f(P4) = 3/2$

Max $3/2$ $\quad$ Min $-\sqrt{2}$

③ $$\int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 1 \, dz \right) dy \right) dx \qquad \boxed{3}$$

$$= \int_0^1 \left(\int_0^{1-x} \left((1-x) - y \right) dy \right) dx$$

$$= \int_0^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) dx$$

$$= \frac{1}{2} \int_0^1 (1 - 2x + x^2) \, dx$$

$$= \frac{1}{2} \left(1 - x^2 + \frac{1}{3} x^3 \right) \Big|_0^1 = \frac{1}{6}$$

④ $$\int_R e^{-x^2} dA = \int_0^1 \left(\int_0^x e^{-x^2} dy \right) dx =$$

$$= \int_0^1 x e^{-x^2} dx = \left| \begin{matrix} x^2 = u \\ 2x\,dx = du \end{matrix} \right|$$

$$= \frac{1}{2} \int_0^1 e^{-u} du = \frac{1}{2} \left(-e^{-u} \right) \Big|_0^1 =$$

$$= \frac{1}{2} \left(-e^{-1} + 1 \right) = \frac{1}{2} \left(1 - \frac{1}{e} \right)$$

⑤ $\int_0^{2\pi} \int_1^4 (r-1) r \, dr \, d\theta =$

$\boxed{4}$

$= 2\pi \int_1^4 (r^2 - r) \, dr =$

$= 2\pi \left(\frac{1}{3} r^3 - \frac{1}{2} r^2 \right) \Big|_1^4 =$

$= 2\pi \left(\frac{1}{3} \underbrace{(64-1)}_{63} - \frac{1}{2} \underbrace{(16-1)}_{15} \right)$

$= 2\pi \left(21 - \frac{15}{2} \right) = \pi (42 - 15) = 27\pi$

Area $16\pi - \pi = 15\pi$

Average $\frac{27\pi}{15\pi} = \frac{27}{15} = \frac{9}{5} = 1.8$

Makes sense since we know that average must be $> 3/2 = 1.5$