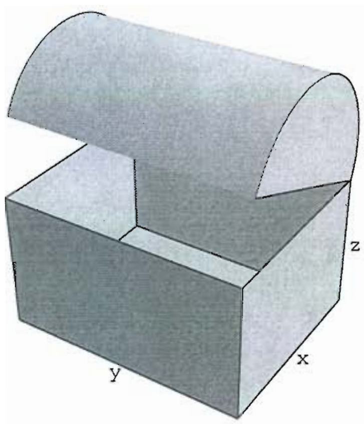
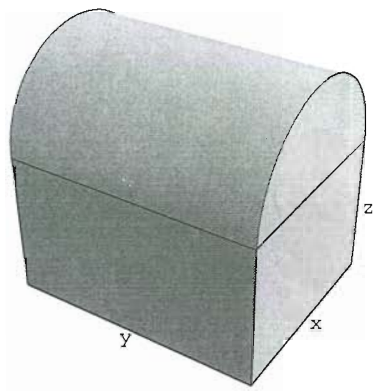
MATH 224 Final Examination December 9, 2009

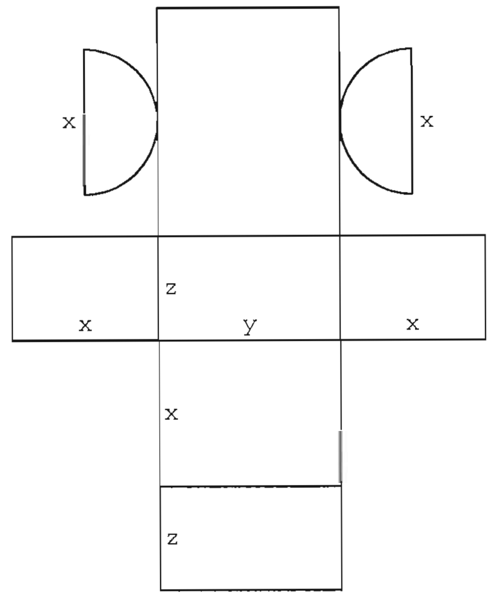
Name Tentative Solutions

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS. There are eight problems. Each is worth 12.5 points.

1. The treasure box pictured below has volume 1. Calculate x, y and z for which the surface area will be minimal.







2. Consider the function $g(x,y) = x^2y + 3y$.
   (a) Find the average rate of change of g as you go from $(3,1)$ to $(0,5)$ along the line segment joining these two points.
   (b) Find the instantaneous rate of change of g as you leave the point $(3,1)$ heading toward $(0,5)$.
   (c) Find the instantaneous rate of change of g as you arrive at the point $(0,5)$ from the direction of $(3,1)$.

3. Captain Astro is in trouble near the sunny side of Mercury. She is at location $(1,1,1)$, and the temperature of the ship's hull when she is at location (x,y,z) will be given by $T(x,y,z) = e^{-x^2-2y^2-2z^2}$, where x, y, and z are measured in meters and the temperature is measured in degrees (these are some special very hot degrees).
   (a) In what direction should she proceed in order to decrease the temperature most rapidly?
   (b) If the ship travels at e^5 meters per second, how fast (in degrees per second) will the temperature decrease if she proceeds in that direction?
   (c) Assume again that the ship travels at e^5 meters per second. Calculate how fast will temperature change if Captain Astro decides to proceed in the direction $\vec{i} + \vec{j} + \vec{k}$. Pay attention to the sign of the change.
   (d) Give one direction in which the rate of change of temperature will be 0.

4. The graph on the right shows three points $A = (2,0,0)$, $B = (0,3,0)$ and $C = (0,0,1)$.
   (a) Use the cross product to find a vector orthogonal to the plane drawn through the points A, B and C.
   (b) Calculate the area of the triangle ABC.
   (c) Calculate the angle in radians between the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$.

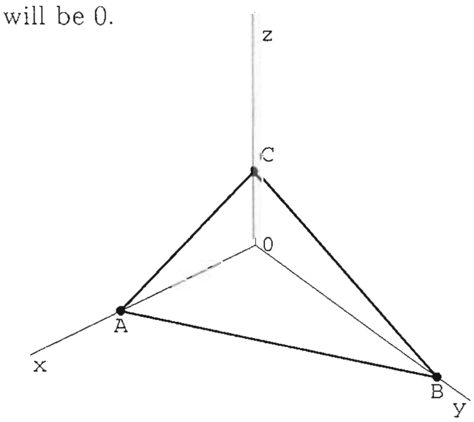


Figure 1: Three points and the plane

5. Use Lagrange multipliers (or any other method) to find the radius of the sphere centered at the origin which touches the plane $8x - 3y - 5z = 14$.

6. The contour diagram in Figure 2 shows contour lines at indicated levels of a function $z = f(x, y)$. Use these contour lines to decide the sign (positive, negative, or zero) of each of the following partial derivatives

$$f_x(P), \qquad f_y(P), \qquad f_{xx}(P),$$
$$f_{yy}(P), \qquad f_{xy}(P), \qquad f_{yx}(P).$$

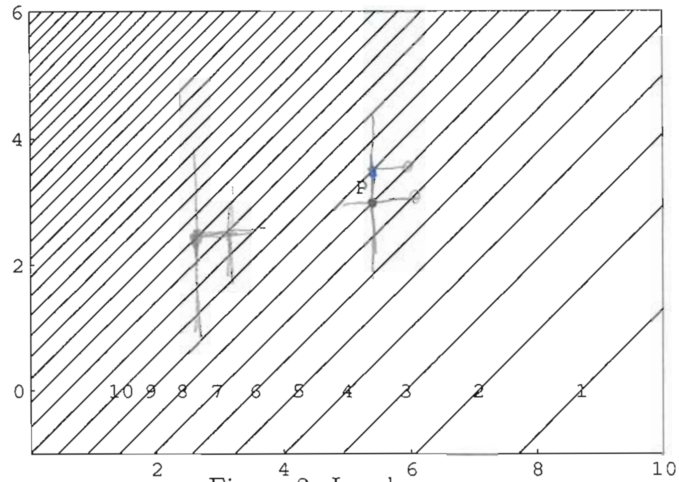


Figure 2: Level curves

7. Consider the following two regions in xy-plane:

$$Q = \{(x, y) : 0 \le x \le y \le 1\}$$
$$R = \{(x, y) : 0 \le y \le x \le 1\}.$$

Consider the following four integrals:

$$I_1 = \int_Q e^x dA, \qquad I_2 = \int_R e^x dA,$$

$$I_3 = \int_Q e^{x^2} dA, \qquad I_4 = \int_R e^{x^2} dA.$$

The goal of this problem is to order the integrals I_1, I_2, I_3, I_4 **from the smallest to the largest.**

(a) Without calculating any integrals determine which one is the smallest and which one is the largest. Briefly explain your reasoning.

(b) Calculate the remaining two integrals. (Find the exact values not approximations.)

(c) List the integrals **from the smallest to the largest.** Briefly explain your reasoning.

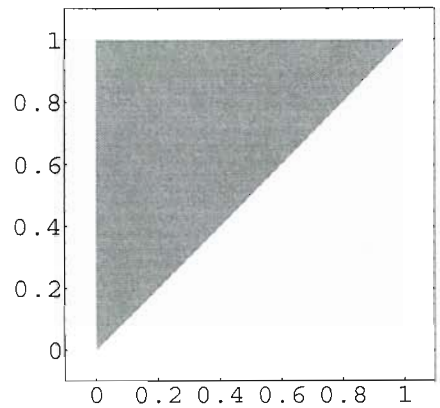


Figure 3: The region Q

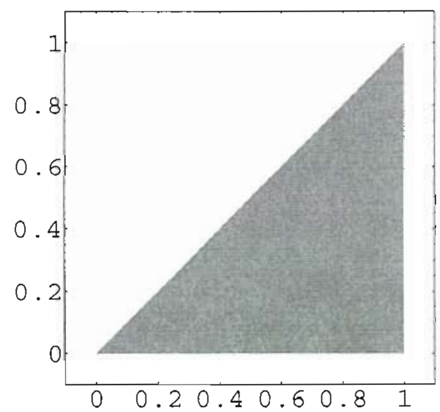


Figure 4: The region R

8. Let W be the piece of the unit sphere centered at the origin which is cut out by the cone $z = \sqrt{x^2 + y^2}$. That is

$$W = \{(x, y, z) : x^2 + y^2 + z^2 \le 1,\ z \ge \sqrt{x^2 + y^2}\}.$$

(a) Use the spherical coordinates to calculate the mass of W assuming that the mass density is constant 1gm/cm^3.

(b) Calculate the exact value for the z-coordinates of the center of mass of W. Explain why the answer must be $> 1/2$.

(1) Volume of a cylinder $\boxed{1}$

$$r^2\pi \cdot h$$

Here we have 1/2 of a cylinder.

$r = x/2$, $h = y$ so volume is

$$\frac{1}{2}\left(\left(\frac{x}{2}\right)^2\pi\right)\cdot y = \frac{1}{8}x^2y\pi$$

So the volume of the treasure box is

$$xyz + \frac{\pi}{8}x^2y = 1$$

Solve for z : $z = \dfrac{1-\frac{\pi}{8}x^2y}{xy} = \dfrac{1}{xy} - \dfrac{\pi}{8}x$

The surface area :

$$A = \underbrace{xy + 2(x+y)z}_{\text{the box}} + \underbrace{\overbrace{\left(\frac{x}{2}\right)^2\pi}^{\text{disk}} + \overbrace{\frac{x}{2}\pi\cdot y}^{\frac{1}{2}\text{cylind.}}}_{\frac{1}{2}\text{ cylinder}}$$

$$A = xy + 2(x+y)z + \frac{x^2}{4}\pi + (xy)\frac{\pi}{2}$$

$$z = \frac{1}{xy} - \frac{\pi}{8}x$$

$$A = xy + 2(x+y)\left(\frac{1}{xy} - \frac{\pi}{8}x\right) + \frac{\pi}{4}x^2 + xy\frac{\pi}{2}$$

$$= xy + \frac{2}{y} + \frac{2}{x} - \frac{\pi}{4}xy - \cancel{\frac{\pi}{4}x^2} + \cancel{\frac{\pi}{4}x^2} + xy\frac{\pi}{2}$$

$$= \left(1+\frac{\pi}{4}\right)xy + \frac{2}{y} + \frac{2}{x} \qquad \boxed{2}$$

$$\frac{dA}{dx} = \frac{4+\pi}{4}\,y - \frac{2}{x^2}$$

$$\frac{dA}{dy} = \frac{4+\pi}{4}\,\cancel{x} - \frac{2}{y^2}$$

$$\rightarrow \quad y = \frac{8}{4+\pi}\,\frac{1}{x^2}$$

$$\frac{4+\pi}{4}\,x = \frac{2}{\left(\frac{8}{4+\pi}\right)^2 \frac{1}{x^4}} = \frac{2x^4}{\left(\frac{8}{4+\pi}\right)^2}$$

$$2x^3 = \left(\frac{\cancel{4+\pi}}{\cancel{4}}\right)\frac{4^2\cdot 2^2}{(4+\pi)^{\cancel{2}}} = \frac{2^2\cdot 4}{4+\pi}$$

$$x^3 = \frac{8}{4+\pi} \qquad x = \frac{2}{\sqrt[3]{4+\pi}}$$

$$y = \frac{8}{4+\pi}\,\frac{(4+\pi)^{2/3}}{4} = \frac{2}{\sqrt[3]{4+\pi}}$$

$$z = \frac{1}{xy} - \frac{\pi}{8} x \qquad \boxed{3}$$

$$z = \frac{1}{\frac{4}{(4+\pi)^{2/3}}} - \frac{\pi}{8} \frac{2}{\sqrt[3]{4+\pi}}$$

$$= \frac{(4+\pi)^{2/3}}{4} - \frac{\pi}{4} \frac{(4+\pi)^{2/3}}{4+\pi}$$

$$= \frac{(4+\pi)^{2/3}}{4} \left(1 - \frac{\pi}{4+\pi} \right) =$$

$$= \frac{(4+\pi)^{2/3}}{\cancel{4}\ 1} \frac{\cancel{4}+\cancel{\pi}-\cancel{\pi}}{4+\pi}$$

$$= \frac{(4+\pi)^{2/3}}{4+\pi} = \frac{1}{\sqrt[3]{4+\pi}}$$

Thus

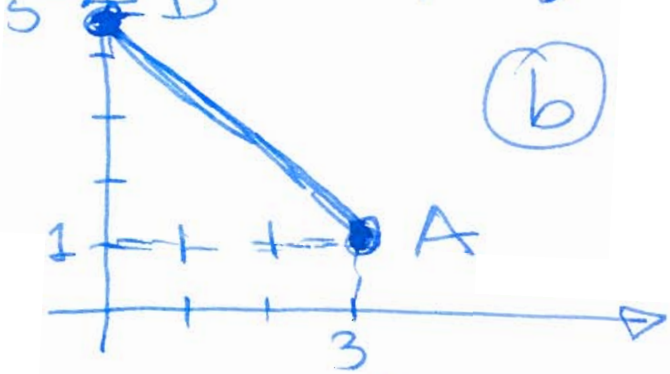
$$\boxed{x = \frac{2}{\sqrt[3]{4+\pi}}, \; y = x, \; z = \frac{1}{2} x}$$

2 $g(x,y) = x^2y + 3y$ $\boxed{4}$

a The average is

$$\frac{g(0,5) - g(3,1)}{\text{distance from } (0,5) \text{ to } (3,1)}$$

$$= \frac{15 - 12}{\sqrt{(-3)^2 + (5-1)^2}} = \frac{3}{5}$$



b

$$\vec{\nabla} g = 2xy\,\vec{i} + (x^2+3)\,\vec{j}$$

b Unit vector of $\overrightarrow{AB}$ is

$$\vec{u} = \frac{1}{5}\left(-3\vec{i} + 4\vec{j}\right)$$

$$= -\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

$$\vec{\nabla} g(3,1) = 6\vec{i} + 12\vec{j} = 6(\vec{i} + 2\vec{j})$$

$$\vec{u} \cdot \vec{\nabla} g(3,1) = 6 \cdot \left(-\frac{3}{5}\right) + 12 \cdot \frac{4}{5}$$

$$= -\frac{18}{5} + \frac{48}{5} = \boxed{6}$$

2 c $\vec{\nabla} g(0,5) = 3\vec{j}$ $\boxed{5}$

$\vec{u} \cdot \vec{\nabla} g(0,5) = \boxed{\frac{12}{5}}$

3 $T(x,y,z) = e^{-x^2-2y^2-2z^2}$

$\vec{\nabla} T(x,y,z) = e^{-x^2-2y^2-2z^2}(-2x\vec{i} - 4y\vec{j} - 4z\vec{k})$

$\vec{\nabla} T(1,1,1) = e^{-5}(-2\vec{i} - 4\vec{j} - 4\vec{k})$

a She should go in the direction

$\boxed{2\vec{i} + 4\vec{j} + 4\vec{k}}$

or unit vector $\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$

b $\|\vec{\nabla} T(1,1,1)\| = 2e^{-5} \cdot 3 = 6e^{-5}$

The temp will decrese at the rate of 6 degrees/sec.

ⓒ The answer is $\boxed{6}$

$$\left(e^5\cdot\right)\left(\vec{\nabla}T(1,1,1)\cdot\left(\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}\right)\right)$$

dot product

$$= e^5\, 2\cdot e^{-5}\left(-\frac{1}{\sqrt{3}}-\frac{2}{\sqrt{3}}-\frac{2}{\sqrt{3}}\right)$$

$$= -2\cdot\frac{5}{\sqrt{3}} = -\frac{10}{\sqrt{3}}\ \underline{\underline{\text{degrees/sec}}}$$

ⓓ Need a direction $\perp$ $\vec{\nabla}T(1,1,1)$

For example

$$2\vec{i}-\vec{j}$$

or any direction $x\vec{i}+y\vec{j}+z\vec{k}$ such that $x+2y+2z=0$.

④ $\vec{AC} = -2\vec{i} + \vec{k}$

$\vec{AB} = -2\vec{i} + 3\vec{j}$

$\boxed{7}$

$$\vec{AC} \times \vec{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix}$$

ⓐ

$$= -3\vec{i} - 2\vec{j} - 6\vec{k}$$

$$\|\vec{AC} \times \vec{AB}\| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

ⓑ

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \cdot \sin\theta$$

θ is the angle between $\vec{a}$ and $\vec{b}$

$$\sin\alpha = \frac{7}{\sqrt{5}\sqrt{13}}$$

$$\alpha = \arcsin\frac{7}{\sqrt{65}}$$

$$\boxed{\cos\alpha = \frac{4}{\sqrt{65}}}$$ Notice $\left(\frac{7}{\sqrt{65}}\right)^2 + \left(\frac{4}{\sqrt{65}}\right)^2 = 1$

5 $\quad 8x-3y-5z=14$ $\boxed{8}$

constraint

minimize $x^2+y^2+z^2$

$$2x=\lambda\cdot 8 \qquad \cancel{x\,z}\ \tfrac{1}{}$$

$$x=4\lambda$$

$$2y=-\lambda\cdot 3 \qquad y=-\tfrac{3}{2}\lambda$$

$$2z=-\lambda\cdot 5 \qquad z=-\tfrac{5}{2}\lambda$$

$$8\cdot 4\lambda+\frac{9}{2}\lambda+\frac{25}{2}\lambda=14$$

$$\lambda=\frac{14\cdot 2}{64+9+25}=\frac{28}{98}=\cancel{\frac{14}{15}}\ \boxed{\frac{2}{7}}$$

So $\quad x^2+y^2+z^2=\cancel{16\left(\frac{14}{15}\right)^2+\frac{9}{4}\lambda^2+\frac{25}{4}\lambda^2}$

$$=\cancel{\left(\frac{14}{15}\right)^2}\left(16+\frac{9}{4}+\frac{25}{4}\right)\left(\frac{2}{7}\right)^2$$

$$=\frac{98}{4}\left(\frac{2}{7}\right)^2=\frac{2\cdot 7^2}{4}\left(\frac{2^2}{7^2}\right)=2$$

the radius is $\sqrt{2}$.

6

$f_x(P) < 0$ with a fixed y f is decr.

$\boxed{9}$

$f_y(P) > 0$ with fixed x f is increasing

$f_{xx}(P) > 0$ with fixed y f look like

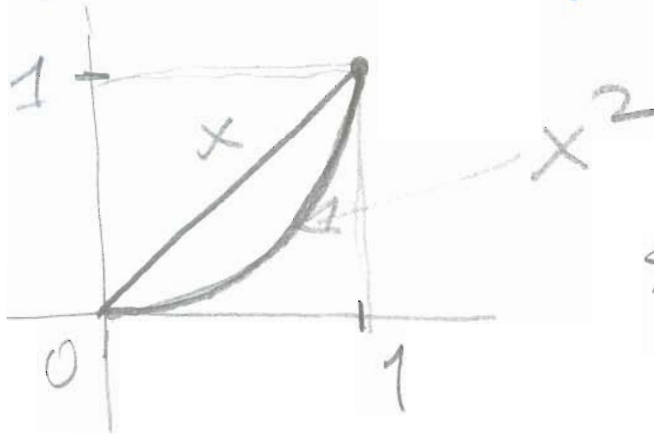
$f_{yy}(P) > 0$ with fixed x f is

$f_{xy}(P) < 0$

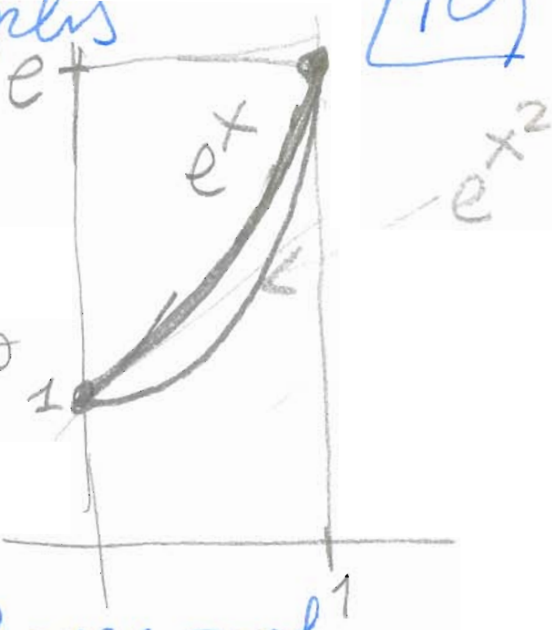
$f_{yx}(P) < 0$

with increasing y $f_x(P)$ becomes more negative: it is decreasing

with increasing x the ~~derivative~~ function f_y becomes a ~~b~~ smaller positive numbe it is decreasing, 7 Notice the graphs $\boxed{10}$



So



a $I_1 > I_3$ lower roof on the same foundation

$I_2 > I_4$

The foundation R has larger area under higher roof. So

$I_2 > I_1$

a $I_4 > I_3$

So I_2 is the largest

I_3 is the smallest

ⓑ calculate I_1 $\boxed{11}$

$$I_1 = \int_Q e^x \, dA = \int_0^1 \int_x^1 e^x \, dy \, dx$$

$$= \int_0^1 \int_0^y e^x \, dx \, dy = \int_0^1 e^x \Big|_0^y dy$$

$$= \int_0^1 (e^y - 1) \, dy = e^y \Big|_0^1 - 1$$

$$= e - 1 - 1 = e - 2$$

$$I_4 = \int_R e^{x^2} dA = \int_0^1 \int_0^x e^{x^2} dy \, dx$$

$$= \int_0^1 x e^{x^2} dx = \left| \begin{array}{l} x^2 = u \\ 2x\,dx = du \end{array} \right|$$

$$= \frac{1}{2} \int_0^1 e^u \, du = \frac{1}{2}(e-1)$$

Since $e < 3 \Rightarrow 2e < e + 3 \Rightarrow$
$2e - 4 < e - 1 \Rightarrow e - 2 < \frac{1}{2}(e-1)$

We have $I_1 < I_4$.

7c finally:

$$I_2 > I_4 > I_1 > I_3$$

8

$$\int_W 1\,dV = m = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \left(\rho^2 \sin\phi\right) d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\pi/4} \frac{1}{3} \sin\phi \, d\phi$$

$$= \frac{2\pi}{3}\left(-\cos\phi\right)\Big|_0^{\pi/4}$$

$$= \frac{2\pi}{3}\left(-\frac{\sqrt{2}}{2} + 1\right)$$

$$= \left(2 - \sqrt{2}\right)\frac{\pi}{3}$$

$$\bar{z} = \frac{1}{m}\int_W z\,dV = \frac{2\pi}{m}\int_0^{\pi/4}\int_0^1 \rho^3 \cos\phi \sin\phi \, d\rho\, d\phi$$

$$= \frac{2\pi}{m}\,\frac{1}{4}\int_0^{\pi/4} \frac{1}{2}\sin 2\phi \, d\phi = \left|\begin{matrix} 2\phi = u \\ 2\,d\phi = du \\ d\phi = \frac{1}{2}du \end{matrix}\right|$$

$\boxed{13}$

$$= \frac{\pi}{2m}\frac{1}{4}\int_0^{\pi/2} \sin u \, du =$$

$$= \frac{\pi}{8m}\left(-\cos u\right)\Big|_0^{\pi/2} = \frac{\pi}{8m}$$

$$= \frac{\pi}{8(2-\sqrt{2})\frac{\pi}{3}} = \frac{3}{8(2-\sqrt{2})}$$

Is $\frac{3}{8(2-\sqrt{2})} > \frac{1}{2}$? Yes

$6 > 16 - 8\sqrt{2}$? Yes

$8\sqrt{2} > 10$? Yes

$4\sqrt{2} > 5$? Yes

$16 \cdot 2 > 25$ Yes