What is the relation between these two uses of surfaces? For example, consider the function

$$f(x, y) = x^2 + y^2 + 3.$$

Define

$$g(x, y, z) = x^2 + y^2 + 3 - z$$

The points on the graph of f satisfy $z = x^2 + y^2 + 3$, so they also satisfy $x^2 + y^2 + 3 - z = 0$. Thus the graph of f is the same as the level surface

$$g(x, y, z) = x^2 + y^2 + 3 - z = 0.$$

In general, we have the following result:

A single surface that is the graph of a two-variable function $f(x, y)$ can be thought of as one member of the family of level surfaces representing the three-variable function

$$g(x, y, z) = f(x, y) - z.$$

The graph of f is the level surface $g = 0$.

Conversely, a single level surface $g(x, y, z) = c$ can be regarded as the graph of a function $f(x, y)$ if it is possible to solve for z. Sometimes the level surface is pieced together from the graphs of two or more two-variable functions. For example, if $g(x, y, z) = x^2 + y^2 + z^2$, then one member of the family of level surfaces is the sphere

$$x^2 + y^2 + z^2 = 1.$$

This equation defines z implicitly as a function of x and y. Solving it gives two functions

$$z = \sqrt{1 - x^2 - y^2} \quad \text{and} \quad z = -\sqrt{1 - x^2 - y^2}.$$

The graph of the first function is the top half of the sphere and the graph of the second function is the bottom half.

Exercises and Problems for Section 12.5

Exercises

1. Match the following functions with the level surfaces in Figure 12.78.

(a) $f(x, y, z) = y^2 + z^2$ **(b)** $h(x, y, z) = x^2 + z^2$.

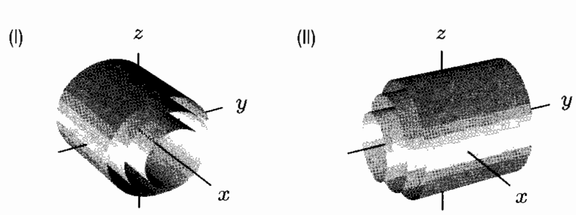

Figure 12.78

2. Match the functions with the level surfaces in Figure 12.79.

(a) $f(x, y, z) = x^2 + y^2 + z^2$
(b) $g(x, y, z) = x^2 + z^2$.

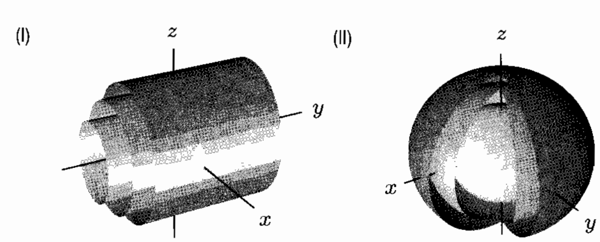

Figure 12.79

3. Write the level surface $x + 2y + 3z = 5$ as the graph of a function $f(x, y)$.
4. Find a formula for a function $f(x, y, z)$ whose level surface $f = 4$ is a sphere of radius 2, centered at the origin.
5. Write the level surface $x^2 + y + \sqrt{z} = 1$ as the graph of a function $f(x, y)$.
6. Find a formula for a function $f(x, y, z)$ whose level surfaces are spheres centered at the point (a, b, c).
7. Which of the graphs in the catalog of surfaces on page 671 is the graph of a function of x and y?

Use the catalog on page 671 to identify the surfaces in Exercises 8–11.

8. $x^2 + y^2 - z = 0$
9. $-x^2 - y^2 + z^2 = 1$
10. $x + y = 1$
11. $x^2 + y^2/4 + z^2 = 1$

In Exercises 12–15, decide if the given level surface can be expressed as the graph of a function, $f(x, y)$.

12. $z - x^2 - 3y^2 = 0$
13. $2x + 3y - 5z - 10 = 0$
14. $x^2 + y^2 + z^2 - 1 = 0$
15. $z^2 = x^2 + 3y^2$

Problems

In Exercises 16–18, represent the surface whose equation is given as the graph of a two-variable function, $f(x, y)$, and as the level surface of a three-variable function, $g(x, y, z) = c$. There are many possible answers.

16. The plane $4x - y - 2z = 6$
17. The top half of the sphere $x^2 + y^2 + z^2 - 10 = 0$
18. The bottom half of the ellipsoid $x^2 + y^2 + z^2/2 = 1$
19. Find a function $f(x, y, z)$ whose level surface $f = 1$ is the graph of the function $g(x, y) = x + 2y$.
20. Find two functions $f(x, y)$ and $g(x, y)$ so that the graphs of both together form the ellipsoid $x^2+y^2/4+z^2/9 = 1$.
21. Find a formula for a function $g(x, y, z)$ whose level surfaces are planes parallel to the plane $z = 2x + 3y - 5$.
22. The surface S is the graph of $f(x, y) = \sqrt{1 - x^2 - y^2}$.
    - **(a)** Explain why S is the upper hemisphere of radius 1, with equator in the xy-plane, centered at the origin.
    - **(b)** Find a level surface $g(x, y, z) = c$ representing S.
23. The surface S is the graph of $f(x, y) = \sqrt{1 - y^2}$.
    - **(a)** Explain why S is the upper half of a circular cylinder of radius 1, centered along the x-axis.
    - **(b)** Find a level surface $g(x, y, z) = c$ representing S.
24. A cone C, with height 1 and radius 1, has its base in the xz-plane and its vertex on the positive y-axis. Find a function $g(x, y, z)$ such that C is part of the level surface $g(x, y, z) = 0$. [Hint: The graph of $f(x, y) = \sqrt{x^2 + y^2}$ is a cone which opens up and has vertex at the origin.]
25. Describe, in words, the level surface $f(x, y, z) = x^2/4 + z^2 = 1$.
26. Describe, in words, the level surface $g(x, y, z) = x^2 + y^2/4 + z^2 = 1$. [Hint: Look at cross-sections with constant x, y and z values.]
27. Describe in words the level surfaces of the function $g(x, y, z) = x + y + z$.
28. Describe in words the level surfaces of $f(x, y, z) = \sin(x + y + z)$.
29. Describe the surface $x^2 + y^2 = (2 + \sin z)^2$. In general, if $f(z) \geq 0$ for all z, describe the surface $x^2 + y^2 = (f(z))^2$.
30. What do the level surfaces of $f(x, y, z) = x^2 - y^2 + z^2$ look like? [Hint: Use cross-sections with y constant instead of cross-sections with z constant.]
31. Describe in words the level surfaces of $g(x, y, z) = e^{-(x^2+y^2+z^2)}$.
32. Sketch and label level surfaces of $h(x, y, z) = e^{z-y}$ for $h = 1, e, e^2$.
33. Sketch and label level surfaces of $f(x, y, z) = 4 - x^2 - y^2 - z^2$ for $f = 0, 1, 2$.
34. Sketch and label level surfaces of $g(x, y, z) = 1 - x^2 - y^2$ for $g = 0, -1, -2$.

12.6 LIMITS AND CONTINUITY

The sheer vertical face of Half Dome, in Yosemite National Park in California, was caused by glacial activity during the Ice Age. (See Figure 12.80.) The height of the terrain rises abruptly by nearly 1000 feet as we scale the rock from the west, whereas it is possible to make a gradual climb to the top from the east.

If we consider the function h giving the height of the terrain above sea level in terms of longitude and latitude, then h has a *discontinuity* along the path at the base of the cliff of Half Dome. Looking at the contour map of the region in Figure 12.81, we see that in most places a small change