MATH 224 Examination 2
February 3, 2012

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
THERE ARE FOUR PROBLEMS. EACH PROBLEM IS WORTH 25 POINTS.

1. Consider the vectors $\vec{v} = 2\vec{\imath} - 2\vec{\jmath} + 1\vec{k}$ and $\vec{w} = -3\vec{\imath} + \vec{\jmath} + 2\vec{k}$.

   (a) Calculate the following three quantities: $\|\vec{v}\|$, $\|\vec{w}\|$ Calculate $\vec{v} \cdot \vec{w}$.

   (b) Write the vector $\vec{w}$ as the sum of two vectors, one parallel and one perpendicular to $\vec{v}$.

(a)

$$\|\vec{v}\| = \sqrt{4+4+1} = 3$$

$$\|\vec{w}\| = \sqrt{9+1+4} = \sqrt{14}$$

$$\vec{v} \cdot \vec{w} = -6-2+2 = -6$$

(b)

$$\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|^2} = \frac{-6}{9} = -\frac{2}{3}$$

$$-\frac{2}{3}\vec{v} = -\frac{4}{3}\vec{\imath} + \frac{4}{3}\vec{\jmath} - \frac{2}{3}\vec{k} \quad \parallel \vec{v}$$

$$\vec{w} - \left(-\frac{4}{3}\vec{\imath} + \frac{4}{3}\vec{\jmath} - \frac{2}{3}\vec{k}\right) =$$

$$= \left(-3+\frac{4}{3}\right)\vec{\imath} + \left(1-\frac{4}{3}\right)\vec{\jmath} + \left(2+\frac{2}{3}\right)\vec{k}$$

$$= -\frac{5}{3}\vec{\imath} - \frac{1}{3}\vec{\jmath} + \frac{8}{3}\vec{k} \quad \perp \vec{v}$$

$*2 \quad *(-2) \quad *1$

$\cancel{-10+2+8=0}$

$$\vec{w} = \underbrace{\left(-\frac{4}{3}\vec{\imath} + \frac{4}{3}\vec{\jmath} - \frac{2}{3}\vec{k}\right)}_{\parallel \vec{v}} + \underbrace{\left(-\frac{5}{3}\vec{\imath} - \frac{1}{3}\vec{\jmath} + \frac{8}{3}\vec{k}\right)}_{\perp \vec{v}}$$

OVER

2. Given three points $A = (3, 4, 2), B = (2, 6, 0)$ and $C = (7, 2, 7)$, find:

   (a) A unit vector which is perpendicular to the plane containing A, B and C.

   (b) The area of the triangle ABC.

   (c) Denote by α the angle at the vertex A in the triangle ABC. Give exact and approximate value for α in radians. (Pay attention here. The answer does not follow directly from (2a).)

   (d) Determine which of the angles $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$ is the closest to the angle α.

a $\quad \vec{AB} = -\vec{i} + 2\vec{j} - 2\vec{k}, \quad \|\vec{AB}\| = 3$

$\vec{AC} = 4\vec{i} - 2\vec{j} + 5\vec{k} \quad \|\vec{AC}\| = \sqrt{45} = 3\sqrt{5}$

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -2 \\ 4 & -2 & 5 \end{vmatrix} = 6\vec{i} - 3\vec{j} - 6\vec{k} = 3(2\vec{i} - \vec{j} - 2\vec{k})$$

$\|\vec{n}\| = 3 * \sqrt{4+1+4} = 9$

$\vec{u} = \frac{2}{3}\vec{i} - \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}$

b $\quad$ ~~area~~ area 9/2

c $\quad \sin\alpha = 9/9\sqrt{5} = 1/\sqrt{5}$

$\vec{AB} \cdot \vec{AC} = -4 - 4 - 10 = -22$

$\cos\alpha = -18/9\sqrt{5} = -2/\sqrt{5}$

$\alpha = \arccos(-2/\sqrt{5}) \approx 2.67795$

d $\quad 5\pi/6$

OVER

3. A contour diagram of a continuous function $z = f(x,y)$ is given in Figure 1. This contour plot is in the xy-plane with the contours labeled by the corresponding z values ranging from 0 to 0.9 in steps of 0.1. Notice that the point $(0.5, 1.5)$ is indicated on the plot. Answer the following questions:

   (a) On a separate plot in xz-plane graph the function $z = f(x, 1.5)$.

   (b) On a separate plot in yz-plane graph the function $z = f(0.5, y)$.

   (c) Based on the contour plot give good estimates of $f_x(0.5, 1.5)$ and $f_y(0.5, 1.5)$. Make sure that your estimates are consistent with the plots you provided in (3a) and (3b).

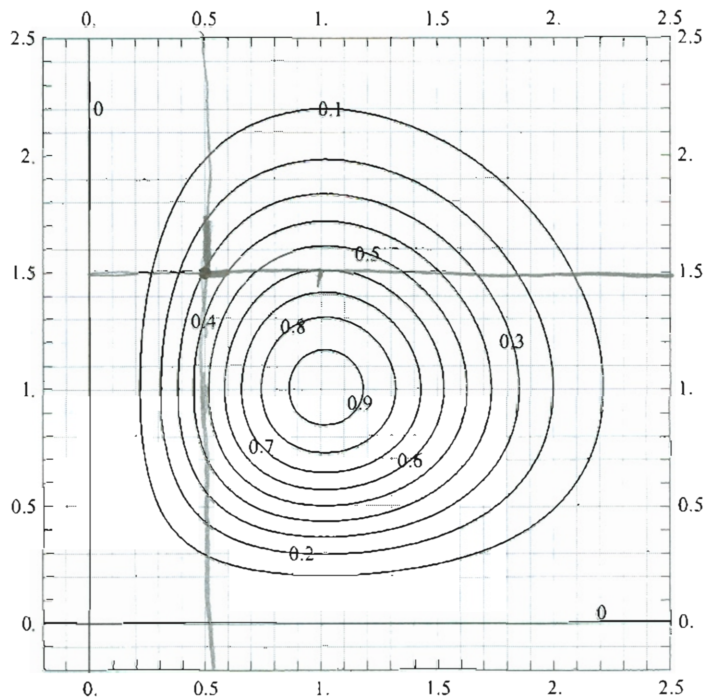
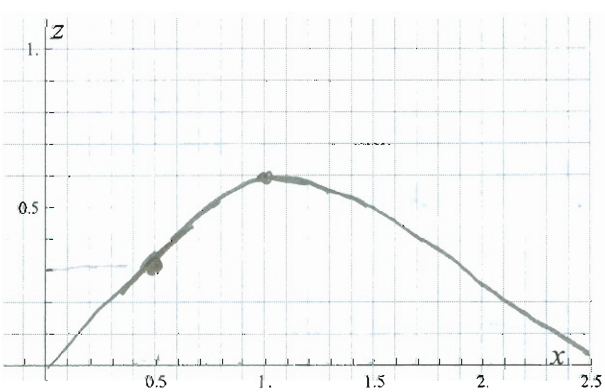
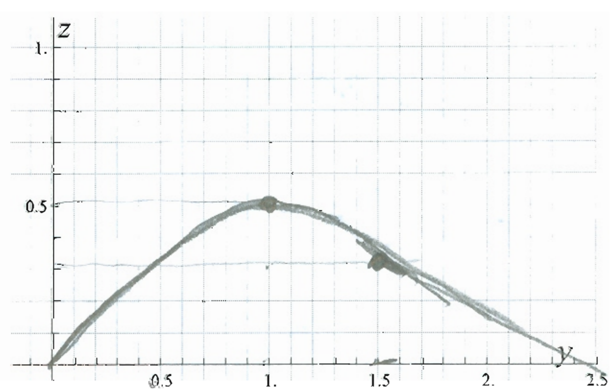


Figure 1: Problem 3





$$f_x(0.5, 1.5) \approx \frac{0.4-0.3}{0.1} = 1$$

$$f_y(0.5, 1.5) \approx \frac{0.2-0.3}{0.2} = -0.5$$

4. In this problem we consider the function $f(x,y) = \sqrt{x^2+y^2}$ and its graph; that is the surface $z = \sqrt{x^2+y^2}$. Notice that $f(3,4) = 5$, that is the point $(3,4,5)$ is on this surface.

   (a) Find the equation of the tangent plane to the graph of the function $f(x,y)$ at the point $(3,4,5)$.

   (b) Show that the tangent plane which you found in (4a) passes through the origin.

   (c) In this item replace the point $(3,4)$ with an arbitrary point $(a,b) \neq (0,0)$. Show that the tangent plane to the graph of the function $f(x,y) = \sqrt{x^2+y^2}$ at the point $\left(a, b, \sqrt{a^2+b^2}\right)$ passes through the origin.

   (d) Using what we learned for the first exam, can you explain why all tangent planes pass through the origin?

(a) $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$ $\quad \dfrac{\partial f}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$

$\left.\dfrac{\partial f}{\partial x}\right|_{(3,4)} = \dfrac{3}{5}$ $\quad \left.\dfrac{\partial f}{\partial y}\right|_{(3,4)} = \dfrac{4}{5}$

$$z = (x-3)\frac{3}{5} + (y-4)\frac{4}{5} + 5 = \frac{3}{5}x + \frac{4}{5}y$$

(b) Clearly $x=0, y=0$ yield $z=0$

(c) $z = (x-a)\dfrac{a}{\sqrt{a^2+b^2}} + (y-b)\dfrac{b}{\sqrt{a^2+b^2}} + \sqrt{a^2+b^2}$

$$= \frac{xa+yb}{\sqrt{a^2+b^2}} - \frac{a^2+b^2}{\sqrt{a^2+b^2}} + \sqrt{a^2+b^2} = \frac{ax+by}{\sqrt{a^2+b^2}}$$

Clearly $x=0, y=0$ yield $z=0$.

(d) The surface is the cone