MATH 224 Final Examination March 14, 2012

Name ______________________

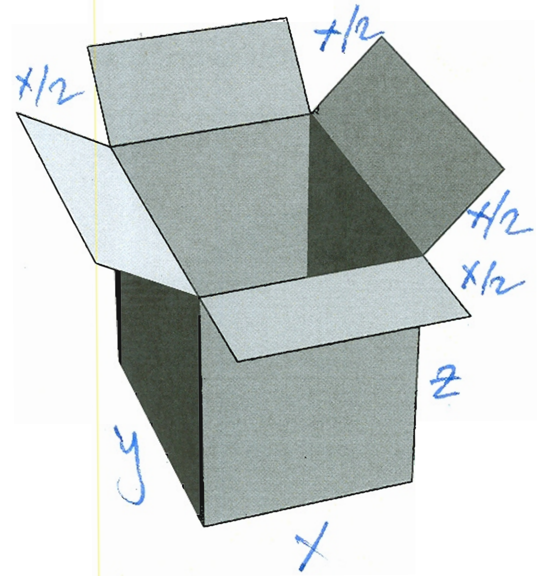
GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS. There are eight problems. Each is worth 12.5 points.

1. The goal of this problem is to design a cardboard box with the volume of 1 cubic unit and with the minimum surface area.

   (a) Label the edges on the picture to the right with x, y, z. Calculate the exact values of x, y, z which give the minimum area.

   (b) Calculate the exact value of the minimum area.

   (Notice that the bottom of the box is constructed in the same way as the top. This is not shown in the picture.)



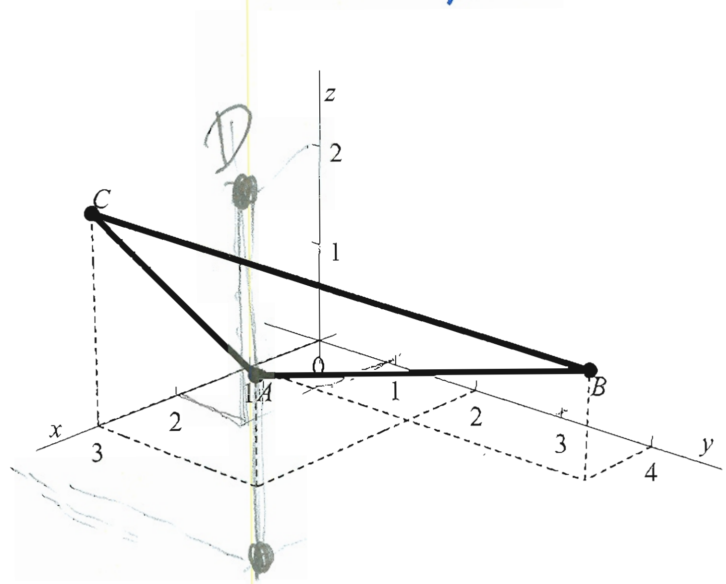
2. The graph on the right shows three points $A = (3,2,1)$, $B = (1,4,1)$ and $C = (3,0,2)$.

   (a) Use the cross product to find a vector orthogonal to the plane drawn through the points A, B and C.

   (b) Calculate the area of the triangle ABC.

   (c) Calculate the angle in radians between the vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$.

   (d) Find a point D in the same plane as A, B, C such that the triangle ABD is a right triangle. Notice that the vectors $\overrightarrow{AD}$ and $\overrightarrow{AB}$ must be orthogonal.



3. The monthly payment P on a mortgage amortized over 30 years at nominal interest rate $r\%$ compounded monthly is a function of two variables: the mortgage amount L and the interest rate r. That is $P = f(L, r)$. In this problem L is given in thousands of dollars, r is given in percentages and P is in dollars.

   (a) Explain the financial significance of the numerical information given below:

   (A) $f(230, 5) = 1,234.69$; (B) $\left.\dfrac{\partial f}{\partial r}\right|_{(L,5)} = 0.61\,L$; (C) $\left.\dfrac{\partial f}{\partial L}\right|_{(L,\cancel{4.5})} = 5.37$ 5

   For each number in (A), (B) and (C) above provide the corresponding units.

   (b) Find a local linearization of the function $f(L, r)$ near the point $(230, 5)$.

   (c) Assume that you plan to borrow between 200 and 240 thousands dollars. Assume also that the interest rate fluctuates between 4.5% and 5.5%. Give an estimate for the lowest and the highest monthly mortgage payment P under these assumptions and using the information given above.

OVER

4. In this problem we consider the region in Oxy plane shaded in gray in Figure 1. Denote this region by R.

   (a) Find the exact average value of the product xy if the point (x, y) belongs the region R.

   (b) Find the exact average value of the sum $x + y$ if the point (x, y) belongs to the region R.

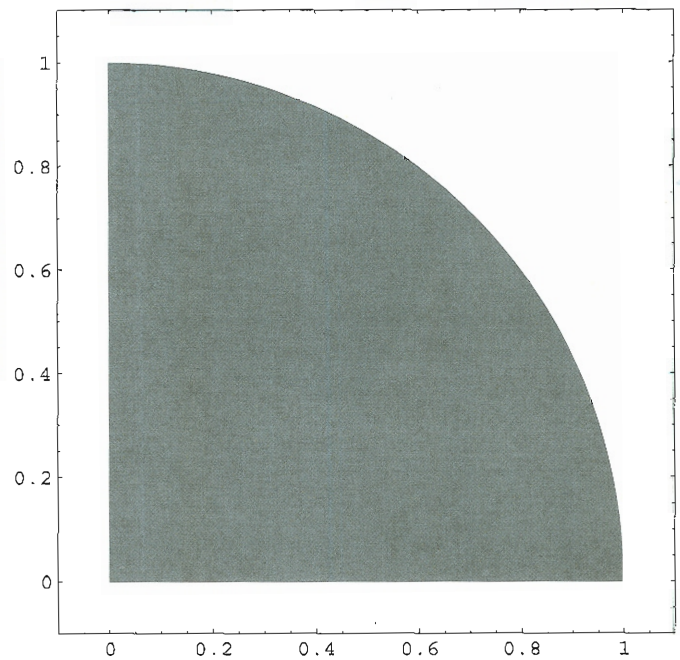


Figure 1: The region R

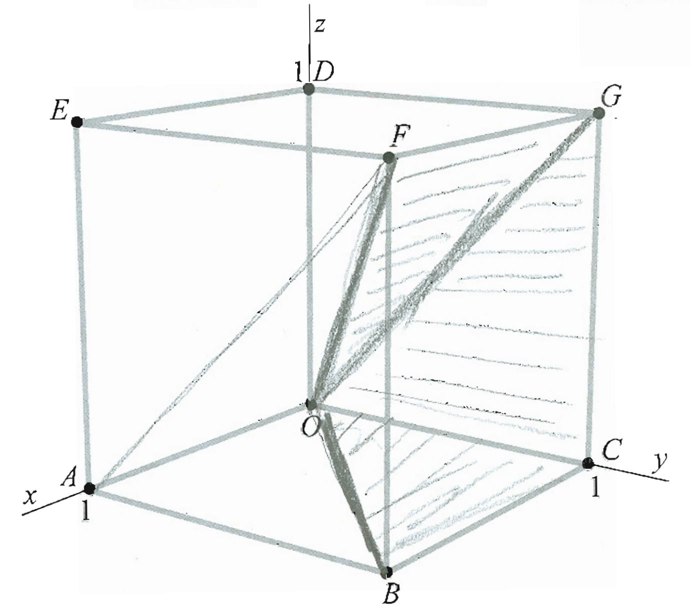


Figure 2: The unit cube

5. Consider the triple integral: $\int_0^1 \int_z^1 \int_0^y z\, dx\, dy\, dz$. Denote by W the region of integration in this integral.

   (a) Evaluate this integral.

   (b) The region W is a pyramid in the unit cube in $Oxyz$ space. In Figure 2 I denoted the vertexes of the unit cube by O, A, B, C, D, E, F, G. Determine which of these eight vertexes are the vertexes of the pyramid W.

6. The temperature at a point (x, y, z) in space is given by

$$T(x, y, z) = 2xyz - 4xy - x^2$$

where T is measured in °C and x, y, z in meters. A bug is located at the point $P = (1, 2, 3)$.

   (a) The bugs leaves the point P on the straight line towards the point $Q = (2, 3, 4)$. At which rate is the temperature changing as the bug leaves the point P?

   (b) In which direction should bug proceed from P in order to increase the temperature most rapidly?

   (c) If the bug travels at 1 meters per second, if it proceeds in the direction that you found in (6b), how fast will the temperature increase? (Denote this rate of increase by r.)

   (d) Find a vector in a direction in which the rate of change of temperature at P is 0.

   (e) Find a vector in a direction in which the rate of change of temperature at P is $r/2$.

7. In this problem we consider the unit sphere and two integrals related to it.

   (a) Use ~~polar~~ spherical coordinates to calculate the volume of the unit sphere. Give all details of your calculation.

   (b) Determine the mass of the unit sphere whose density at the point P is given as the distance of the point P from the z-axis. To help you visualize this unit sphere, in Figure 3 I provided a cross section of this sphere and the xz-plane.

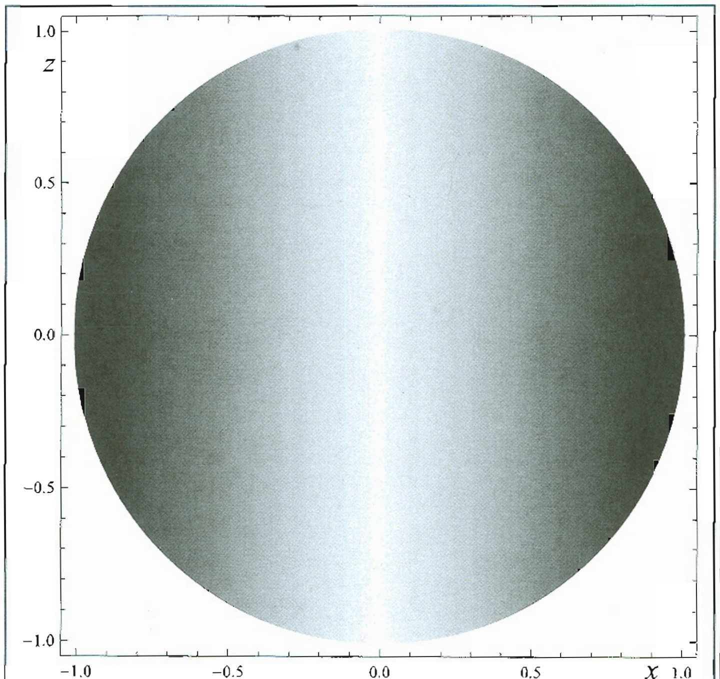


Figure 3: The unit sphere

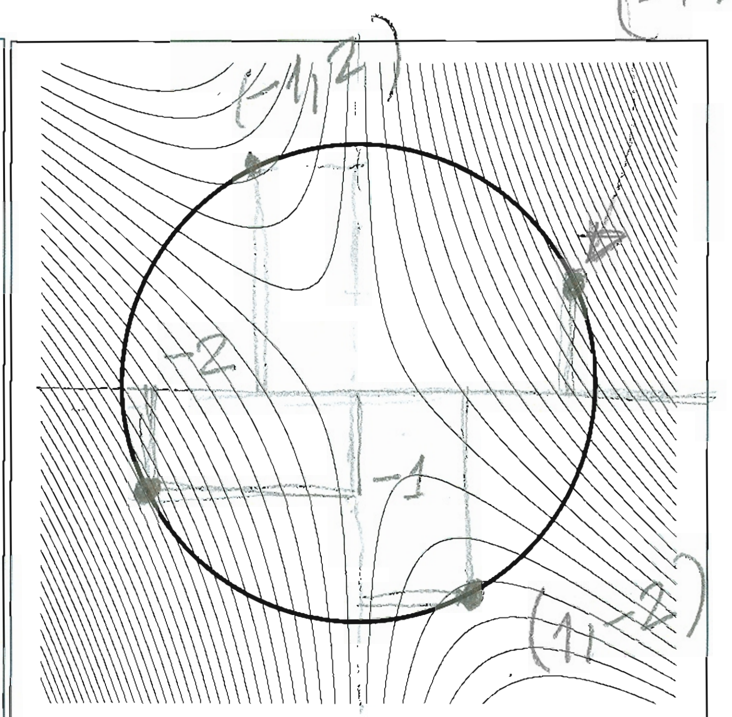


Figure 4: For Problem 8

8. Consider the function

$$f(x,y) = 3x^2 + 4xy + 1$$

subject to the constraint

$$x^2 + y^2 = 5.$$

   (a) Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint. (Hint: The values of the parameter λ are -1 and 4. To help you with this problem, in Figure 4 I provided a contour plot of f and the graph of the constraint.)

   (b) If the constraint is changed to

$$x^2 + y^2 = 5.1,$$

   estimate, without any calculations, the maximum and minimum values of f subject to this new constraint.

① ⓐ The surface area is

$2(x+y)(z+x)$. $\boxed{1}$

We can minimize the half of the area.

~~S(x,y~~ $z = 1/xy$

$$S(x,y) = (x+y)\left(\frac{1}{xy} + x\right)$$

$$= \frac{1}{y} + \frac{1}{x} + x^2 + xy$$

$$S_x = -\frac{1}{x^2} + 2x + y = 0$$

$$S_y = -\frac{1}{y^2} + x = 0. \quad \text{So } x = \frac{1}{y^2}$$

$$\boxed{-y^4 + \frac{2}{y^2} + y = 0} \quad \text{since } y \neq 0$$

$$-y^3 + \frac{2}{y^3} + 1 = 0. \quad \text{Set } y^3 = w$$

$$w = \frac{2}{w} + 1, \text{ so } w^2 - w - 2 = 0$$

$$w_{1,2} = \frac{1 \pm 3}{2}$$

$$\underline{w_1 = 2}, \quad \cancel{\frac{w_2 = -1}{y > 0}}$$

So $y = \sqrt[3]{2}$

$$x = \frac{1}{\sqrt[3]{4}} = \frac{1}{2}\sqrt[3]{2}$$

~~zA~~ $xy = \frac{1}{2}\sqrt[3]{4} = \frac{1}{\sqrt[3]{2}}$

$z = \sqrt[3]{2} = y$

1 b The surface area

$$2(x+y)(z+x) = 2(x+y)^2$$

$$= 2\left(\frac{3}{2}\sqrt[3]{2}\right)^2 = \underline{\underline{\frac{9}{2}\sqrt[3]{4}}}$$

2 a

$$\vec{AB} = -2\vec{i} + 2\vec{j} + 0\vec{k}$$

$$\vec{AC} = 0\vec{i} - 2\vec{j} + \vec{k}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 0 \\ 0 & -2 & 1 \end{vmatrix} = (2)\vec{i} - (-2)\vec{j} + 4\vec{k}$$

$$= 2(\vec{i} + \vec{j} + 2\vec{k})$$

vector $\perp$ to the plane $\vec{i} + \vec{j} + 2\vec{k}$

b area of $\triangle ABC$ is $\sqrt{6}$

c $\|\vec{AB} \times \vec{AC}\| = 2\sqrt{6} = \|\vec{AB}\|\,\|\vec{AC}\|\sin\alpha$

$$\vec{AB} \cdot \vec{AC} = -4 = \|\vec{AB}\|\,\|\vec{AC}\|\cos\alpha$$

$\sin\alpha = \sqrt{\frac{3}{5}}$, $\cos\alpha = -\sqrt{\frac{2}{5}}$

So $\alpha = -\arcsin\sqrt{\frac{3}{5}} = \arccos\left(-\sqrt{\frac{2}{5}}\right)$

$$= \pi - \arctan\sqrt{\frac{3}{2}}.$$

2 d

$$\vec{n} = \vec{i} + \vec{j} + 2\vec{k}$$

$\boxed{3}$

$$\frac{1}{2}\overrightarrow{AB} = -\vec{i} + \vec{j}$$

$$\vec{n} \times \left(\frac{1}{2}\overrightarrow{AB}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = (-2)\vec{i} - (2)\vec{j} + 2\vec{k}$$

$$= -2(\vec{i} + \vec{j} - \vec{k})$$

$$\overrightarrow{OD} = \overrightarrow{OA} + (\vec{i} + \vec{j} - \vec{k})$$

$$= 4\vec{i} + 3\vec{j}, \quad D = (4,3,0)$$

or

$$\overrightarrow{OD} = \overrightarrow{OA} - (\vec{i} + \vec{j} - \vec{k}) = 2\vec{i} + \vec{j} + 2\vec{k}$$

$$D = (2,1,2)$$

3 a

(A) gives monthly payment in \$

(B) is in \$/%; it tells how much will monthly pmt increase for each % increase in interest rate.

(C) is in \$/\$1K; it tells how much will monthly pmt increase for each \$1,000 increase in the loan amount.

③ ⓑ

$\boxed{4}$

$$f(L,r) \approx 1{,}234.69 + 0.61\,\cancel{\ast}^{\,230}(r-5) + 5.37(L-230)$$

ⓒ

$$\begin{matrix}\max \\ \min\end{matrix}\; 1{,}234.69 \pm 0.5 * 0.61 * 230 \pm 5.37 * 10$$

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④ⓐ The area of R is $\pi/4$.

$$\frac{4}{\pi}\iint_R xy\,dA = \frac{4}{\pi}\int_0^{\pi/2}\int_0^1 r\cos\theta\, r\sin\theta\; r\,dr\,d\theta$$

$$= \frac{4}{\pi}\,\frac{1}{4}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta = \left|\begin{array}{l} \sin\theta = u \\ \cos\theta\,d\theta = du \\ \begin{array}{c|c} \theta & u \\ \hline 0 & 0 \\ \pi/2 & 1 \end{array}\end{array}\right|$$

$$= \frac{1}{\pi}\int_0^1 u\,du = \frac{1}{2\pi}$$

The average product is $\frac{1}{2\pi}$.

Compare to $\square$ $\iint xy\,dy = $ $\square$ $1/4$

5

4 b

$$\frac{4}{\pi}\int_0^{\pi/2}\int_0^1 r(\cos\theta+\sin\theta)\,r\,dr\,d\theta$$

$$=\frac{4}{\pi}\,\frac{1}{3}\int_0^{\pi/2}(\cos\theta+\sin\theta)\,d\theta=\frac{8}{3\pi}$$

Compare to $\iint_{\square}(x+y)\,dA = 1$

larger ok.

5 a

$$\int_0^1\int_z^1\int_0^y z\,dx\,dy\,dz=\int_0^1\int_z^1 yz\,dy\,dz$$

$$=\int_0^1 z\,\frac{1}{2}y^2\Big|_z^1\,dz=\frac{1}{2}\int_0^1 z(1-z^2)\,dz$$

$$=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)=1/8$$

5 b The region W is the pyramid

$OBCGF$.

The volume of this pyramid is $1/3$.

In fact we calculate the mass when density = z

6 (a)

$$\overrightarrow{PQ} = \vec{i} + \vec{j} + k$$

$$\vec{u} = \frac{1}{\sqrt{3}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

$\boxed{6}$

$$\left(\vec{\nabla} T\right)(x, y, z) = \left(2\overset{2}{y}\overset{3}{z} - 4\overset{2}{y} - 2\overset{1}{x}\right)\vec{i}$$
$$+ \left(2\overset{1}{x}\overset{3}{z} - 4\overset{1}{x}\right)\vec{j}$$
$$+ 2\overset{1}{x}\overset{2}{y}\,\vec{k}$$

$$\left(\vec{\nabla} T\right)(P) = 2\vec{i} + 2\vec{j} + 4\vec{k}$$

$$\left(\vec{\nabla} T\right)(P) \cdot \vec{u} = \frac{8}{\sqrt{3}} \quad {}^\circ C/\text{meter}$$

(b) Proceed in direction $\vec{i} + \vec{j} + 2\vec{k}$

(c) $r = 2\sqrt{6} = \left\| \left(\vec{\nabla} T\right)(P) \right\|$

(d) $\vec{i} - \vec{j} + 0\vec{k}$ is one direction in which no change.

(e) unit vector

$$\vec{v} = \frac{1}{2}\frac{1}{\sqrt{6}}\left(\vec{i} + \vec{j} + 2\vec{k}\right) + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\left(\vec{i} - \vec{j}\right).$$

7 a

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3} \sin\phi \, d\phi \, d\theta =$$

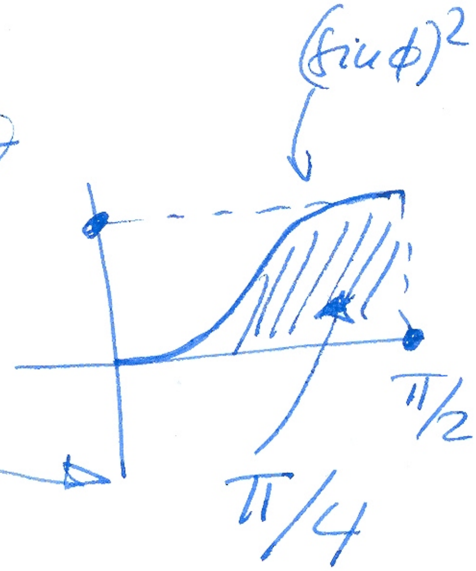
$$= \frac{1}{3}\int_0^{2\pi}\left(\int_0^{\pi} \sin\phi \, d\phi\right) d\theta =$$

$$= \frac{2}{3}\int_0^{2\pi} d\theta = \boxed{\frac{4\pi}{3}}$$

b

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho \sin\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\underbrace{\int_0^{\pi} (\sin\phi)^2 \, d\rho}\, d\theta$$



$$= \frac{\pi}{16}\int_0^{2\pi} d\theta = \frac{\pi^2}{\cancel{8}\,4}$$

8 (a) $\vec{\nabla} f = (6x+4y)\vec{i} + 4x\vec{j}$

$\boxed{8}$

$\vec{\nabla} g = 2x\vec{i} + 2y\vec{j}$

$6x+4y=2\lambda x$	$3x+2y=\lambda x$
$4x = 2\lambda y$	$2x = \lambda y$
$x^2+y^2=5$	$x^2+y^2=5$

$y = \frac{\lambda-3}{2}x$, $y = \frac{2}{\lambda}x$

so $\frac{\lambda-3}{2} = \frac{2}{\lambda} \Rightarrow \lambda^2-3\lambda-4=0$

$\lambda_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} = 4$ or -1

If $\boxed{\lambda = 4}$, then $x=2y$, so $4y^2+y^2=5$

$y = \pm 1$, $x = \pm 2$ Points $(2,1)$ & $(-2,-1)$

Values $f(2,1) = f(-2,-1) = 21$ max

If $\lambda=-1$, then $y=-2x$ so $x^2+4x^2=5$.

$x = \pm 1$, $y = \mp 2$ Points $(1,-2)$ & $(-1,2)$

Values $f(1,-2) = f(-1,2) = -4$ min

(b) max ≈ 21.4 min ≈ -4.1.