Problem 2 posted on January 26

I did this problem in class. The parametric equation of the curve that we are studying is

$$\left\{x, \frac{1}{4} x^2 - \frac{1}{2} \text{Log}[x]\right\}$$

So the velocity vector is

```
In[10]:= D[{x, 1/4 x^2 - 1/2 Log[x]}, x]
```

$$\text{Out[10]}= \left\{1, -\frac{1}{2\,x} + \frac{x}{2}\right\}$$

Thus the magnitude of the velocity vector is

```
In[11]:= Sqrt[{1, -1/(2 x) + x/2}.{1, -1/(2 x) + x/2}]
```

$$\text{Out[11]}= \sqrt{1 + \left(-\frac{1}{2\,x} + \frac{x}{2}\right)^2}$$

To find the asked arc length we integrate

```
In[13]:= Integrate[Sqrt[1 + (-1/(2 x) + x/2)^2], {x, 1, E}]
```

$$\text{Out[13]}= \frac{1}{4}\left(1 + e^2\right)$$

To calculate the last integral by hand we would need to find

```
In[14]:= Integrate[Sqrt[1 + (-1/(2 x) + x/2)^2], x]
```

$$\text{Out[14]}= \frac{x\sqrt{2 + \frac{1}{x^2} + x^2}\ \left(x^2 + 2\,\text{Log}[x]\right)}{4\left(1 + x^2\right)}$$

This looks like a hard integral since *Mathematica* does not realize that the expression that we are integrating is very simple for x > 0:

```
In[16]:= FullSimplify[Sqrt[1 + (-1/(2 x) + x/2)^2], x > 0]
```

$$\text{Out[16]}= \frac{1}{2}\sqrt{2 + \frac{1}{x^2} + x^2}$$

Mathematica does not realize that

```
In[17]:= FullSimplify[Sqrt[1 + (-1/(2 x) + x/2)^2] == 1/2 (x + 1/x), x > 0]

Out[17]= True
```

Thus instead of integrating

```
Integrate[Sqrt[1 + (-1/(2 x) + x/2)^2], x]
```

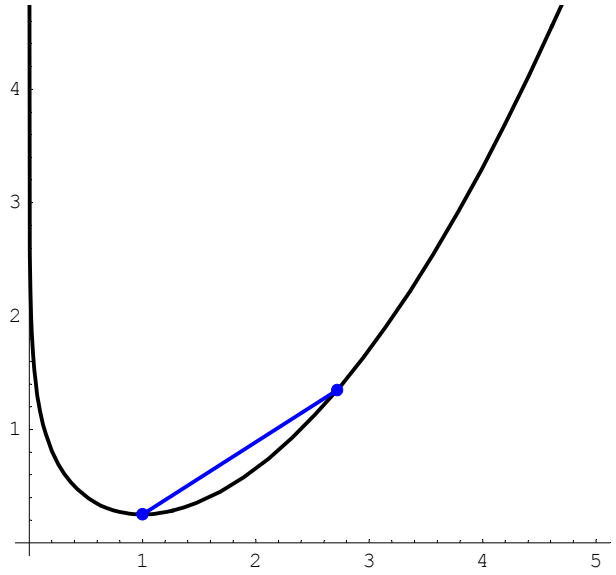
we can integrate

```
In[18]:= Integrate[1/2 (x + 1/x), x]

Out[18]= x^2/4 + Log[x]/2
```

which can be done by hand easily.

Let us illustrate what we have calculated

```
In[22]:= Plot[1/4 x^2 - 1/2 Log[x], {x, 0, 5}, PlotStyle → {Thickness[0.007]},
  Epilog → {Blue, Thickness[0.007], Line[{{1, 1/4}, {E, 1/4 (E^2 - 2)}}], PointSize[0.02],
    Point[{1, 1/4}], Point[{E, 1/4 (E^2 - 2)}]}, AspectRatio → Automatic]
```




```
Out[22]= - Graphics -
```

We calculated the arc length of the black curve between the blue points. Let us calculate the length of the blue line segment between blue points.

```
In[23]:= N[Sqrt[(E - 1)^2 + ((1/4) (E^2 - 2) - 1/4)^2]]
```

```
Out[23]= 2.03874
```

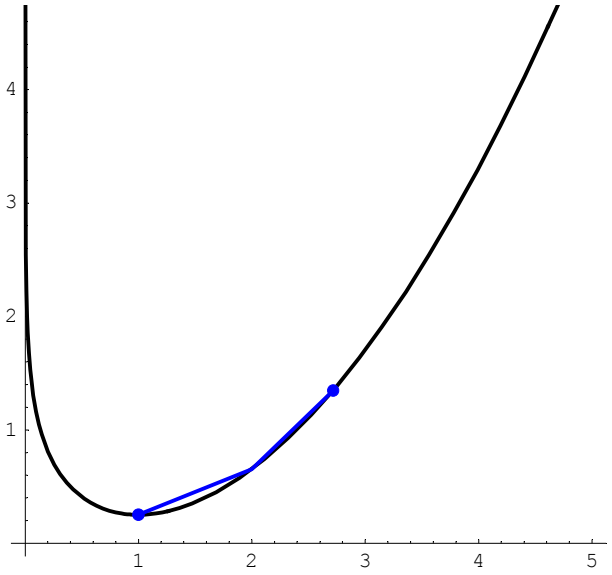
```
In[24]:= N[(1/4) (1 + e^2)]
```

```
Out[24]= 2.09726
```

It is hard to believe that the blue line segment is approx. 2.039 long and the corresponding black arc length on the graph is 2.097.

Let me add one more point.

```
In[26]:= Plot[(1/4) x^2 - (1/2) Log[x], {x, 0, 5}, PlotStyle → {Thickness[0.007]}, Epilog →
           {Blue, Thickness[0.007], Line[{{1, 1/4}, {2, (1/4) (2^2 - 2 Log[2])}, {E, (1/4) (E^2 - 2)}}],
            PointSize[0.02], Point[{1, 1/4}], Point[{E, (1/4) (E^2 - 2)}]}, AspectRatio → Automatic]
```




```
Out[26]= - Graphics -
```

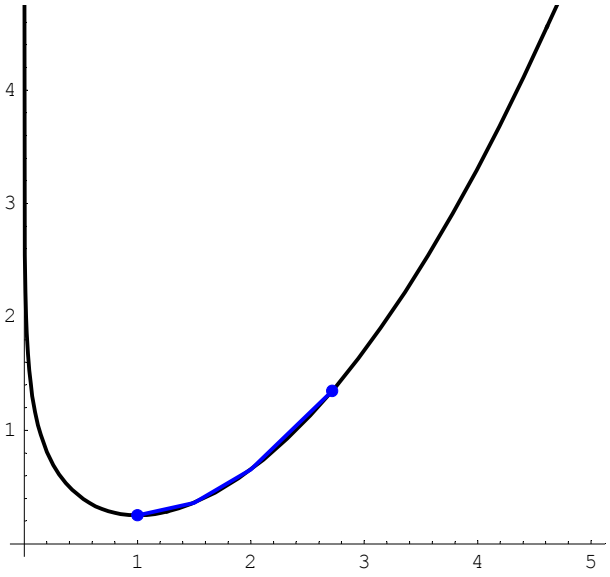
and find the length of the blue polygonal line

```
In[4]:= N[Sqrt[(2 - 1)^2 + ((1/4) (2^2 - 2 Log[2]) - 1/4)^2]] +
        N[Sqrt[(E - 2)^2 + ((1/4) (E^2 - 2 Log[E]) - (1/4) (2^2 - 2 Log[2]))^2]]
```

```
Out[4]= 2.07698
```

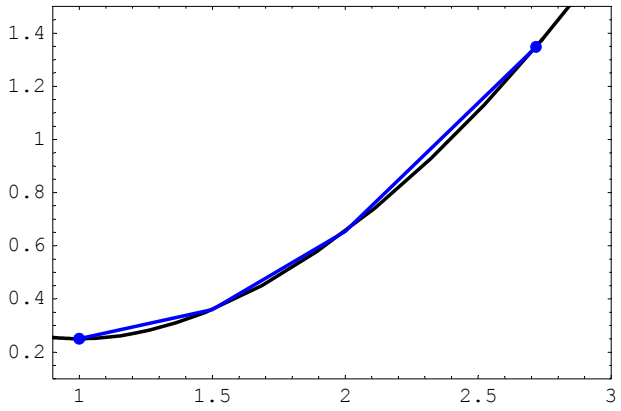
This is less than the arc length and closer to the value that we found for the arc length. This makes it believable that our result is correct. However, we could add one more point

```
In[27]:= Plot[1/4 x^2 - 1/2 Log[x], {x, 0, 5},
  PlotStyle → {Thickness[0.007]}, Epilog → {Blue, Thickness[0.007],
    Line[{{1, 1/4}, {3/2, 1/4 ((3/2)^2 - 2 Log[3/2])}, {2, 1/4 (2^2 - 2 Log[2])}, {E, 1/4 (E^2 - 2)}}],
    PointSize[0.02], Point[{1, 1/4}], Point[{E, 1/4 (E^2 - 2)}]}, AspectRatio → Automatic]
```




```
Out[27]= - Graphics -
```

```
In[30]:= Plot[1/4 x^2 - 1/2 Log[x], {x, 0, 5},
  PlotStyle → {Thickness[0.007]}, Epilog → {Blue, Thickness[0.007],
    Line[{{1, 1/4}, {3/2, 1/4 ((3/2)^2 - 2 Log[3/2])}, {2, 1/4 (2^2 - 2 Log[2])}, {E, 1/4 (E^2 - 2)}}],
    PointSize[0.02], Point[{1, 1/4}], Point[{E, 1/4 (E^2 - 2)}]},
  AspectRatio → Automatic, PlotRange → {{.9, 3}, {.1, 1.5}}, Axes → False, Frame → True]
```




```
Out[30]= - Graphics -
```

The length of the blue polygonal line is

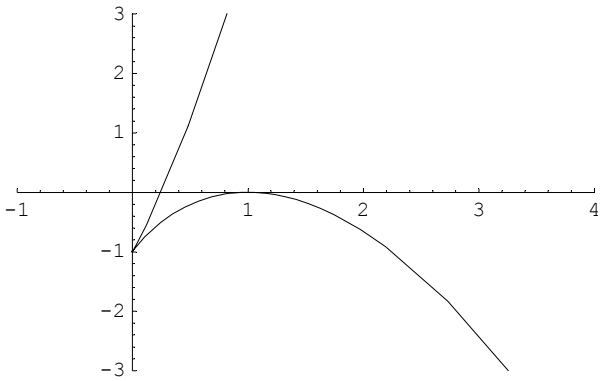
```
In[6]:= N[Sqrt[(3/2 - 1)^2 + (1/4 ((3/2)^2 - 2 Log[3/2]) - 1/4)^2]] +
  N[Sqrt[(2 - 3/2)^2 + (1/4 ((3/2)^2 - 2 Log[3/2]) - 1/4 (2^2 - 2 Log[2]))^2]] +
  N[Sqrt[(E - 2)^2 + (1/4 (E^2 - 2 Log[E]) - 1/4 (2^2 - 2 Log[2]))^2]]

Out[6]= 2.09043

In[7]:= N[1/4 (1 + e^2)]

Out[7]= 2.09726
```

This makes our result even more believable.

Exercise 10 in 17.2

It is convenient to observe that the parametric curve in this exercise is situated in one plane y = 2. So I can plot this curve

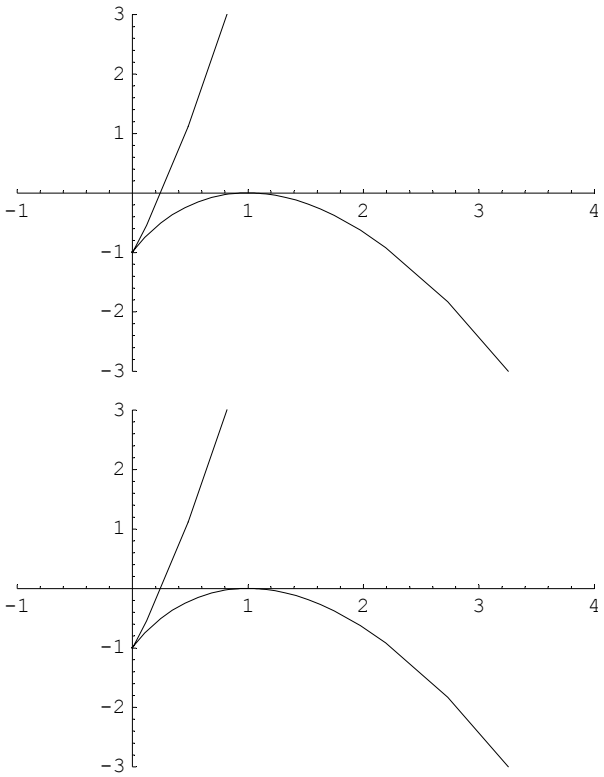
```
In[31]:= ParametricPlot[{(t - 1)^2, 2 t^3 - 3 t^2}, {t, -4, 4}, PlotRange → {{-1, 4}, {-3, 3}}]
```

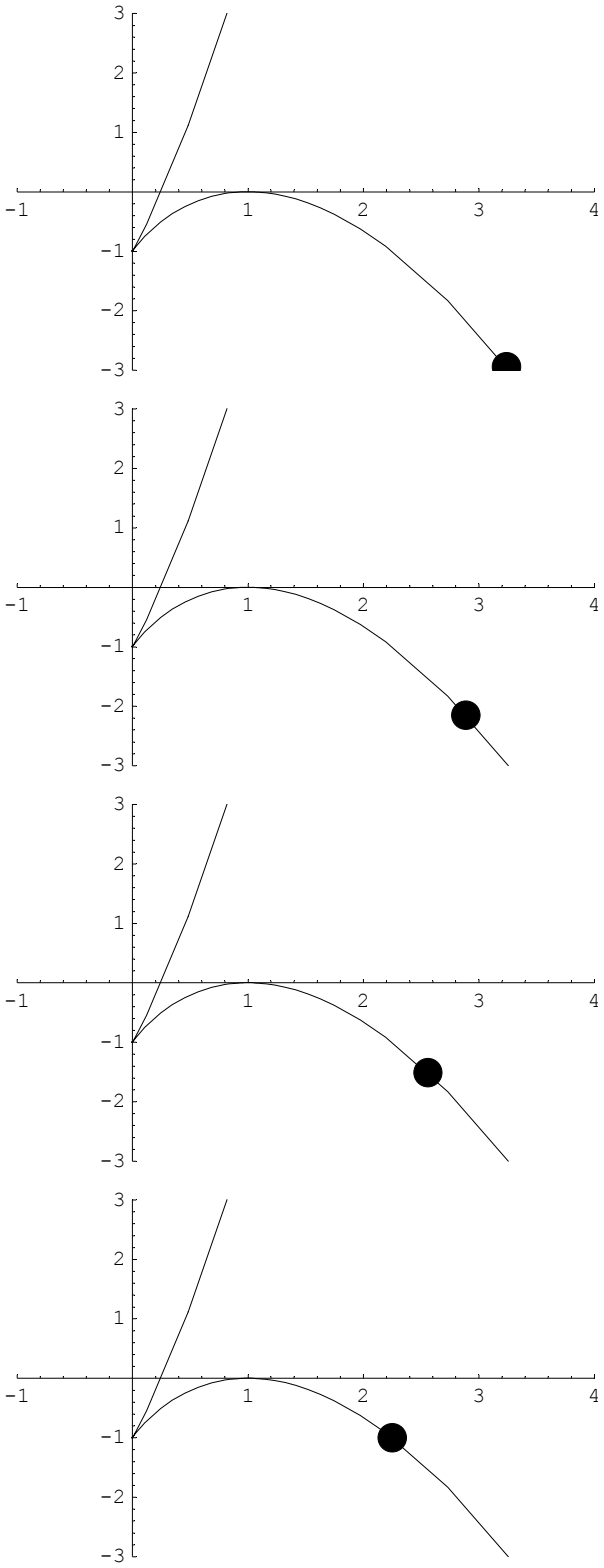


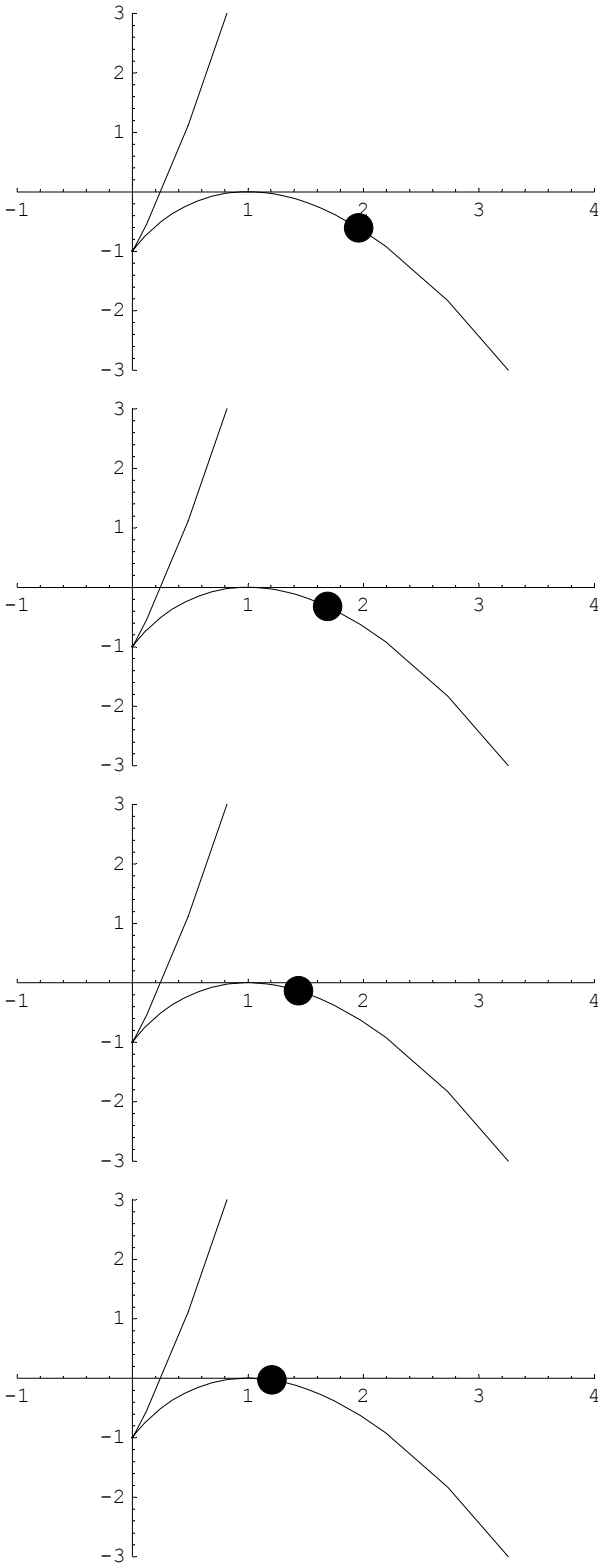

```
Out[31]= - Graphics -
```

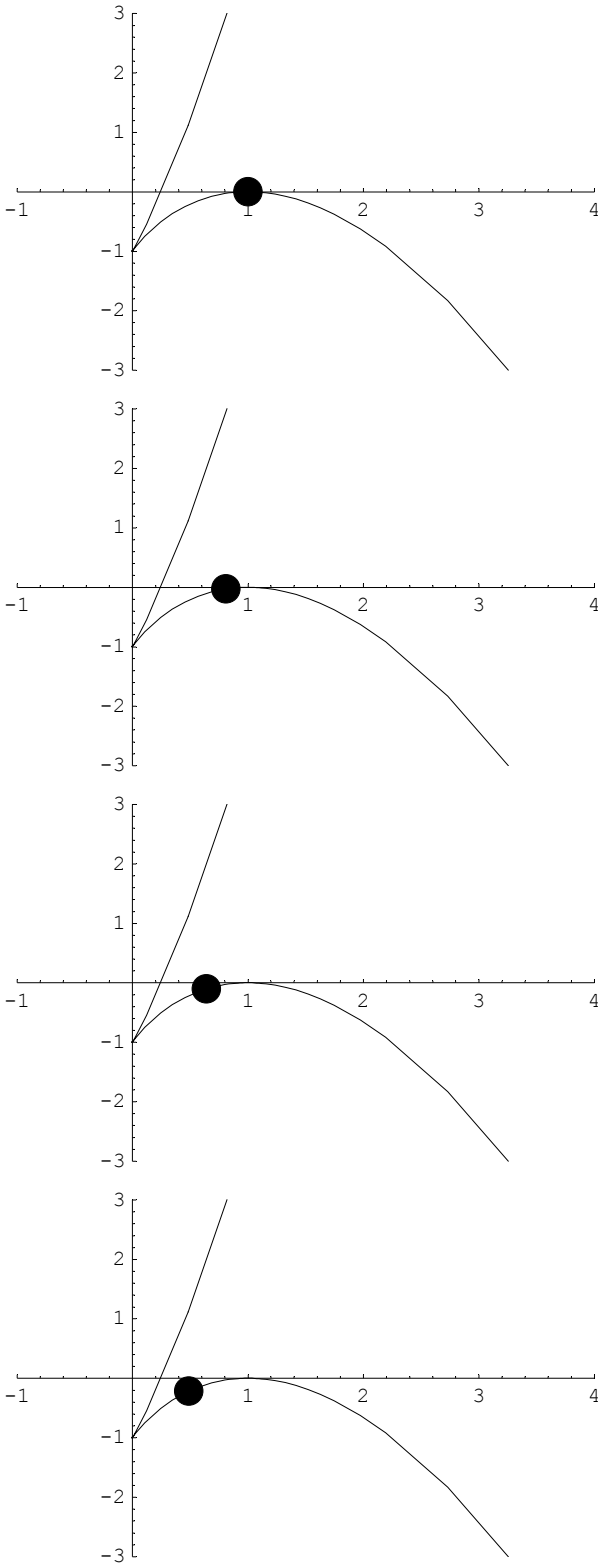
This plot does not tell us much about the motion. We need to think about how a point moves following the given parametric equations. The next animation illustrates that

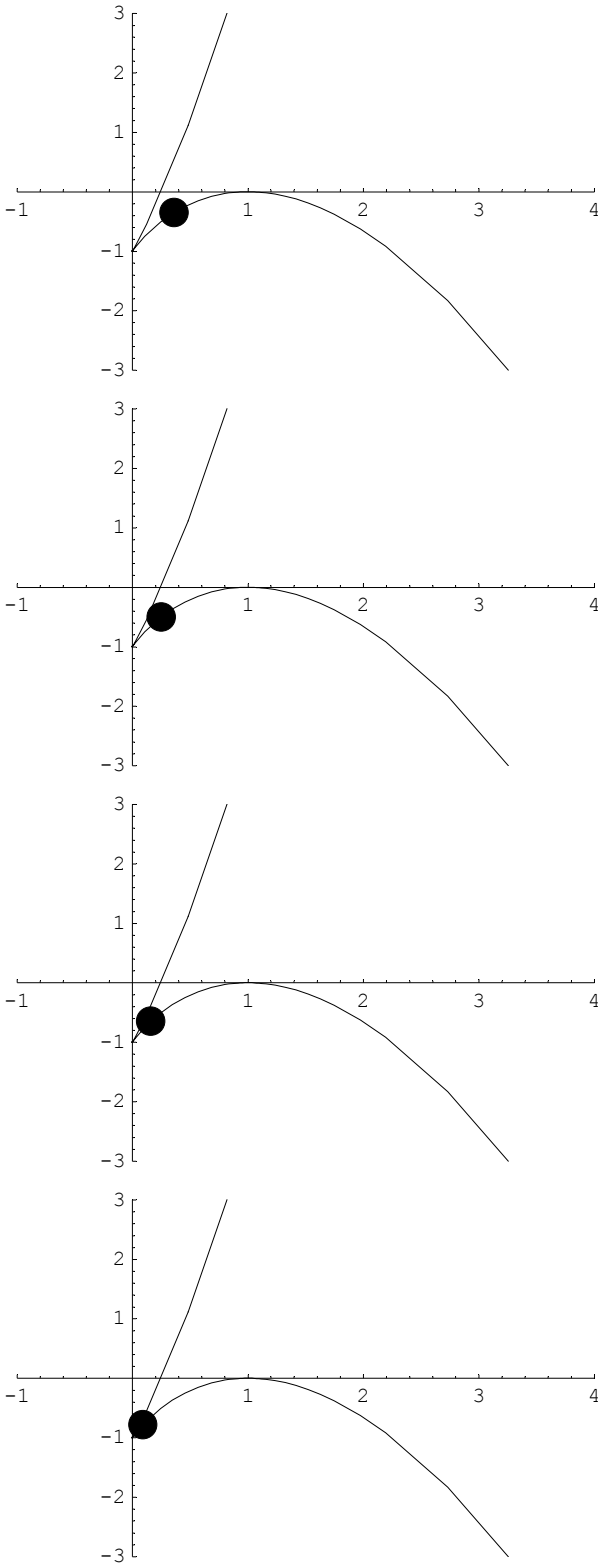
```
In[33]:= Clear[s];
         Table[ParametricPlot[{(t - 1)^2, 2 t^3 - 3 t^2}, {t, -4, 4}, PlotRange → {{-1, 4}, {-3, 3}},
           Epilog → {{PointSize[0.05], Point[{(s - 1)^2, 2 s^3 - 3 s^2}]}}], {s, -1, 2, .1}];
```

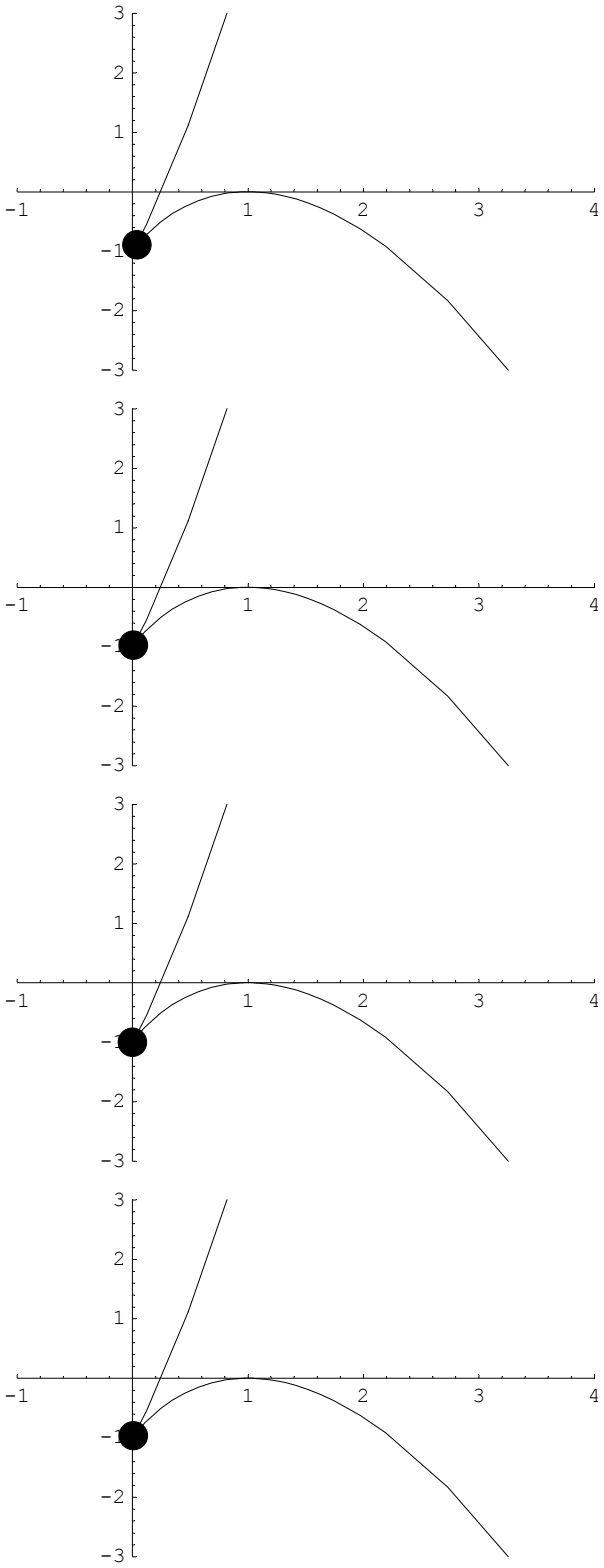


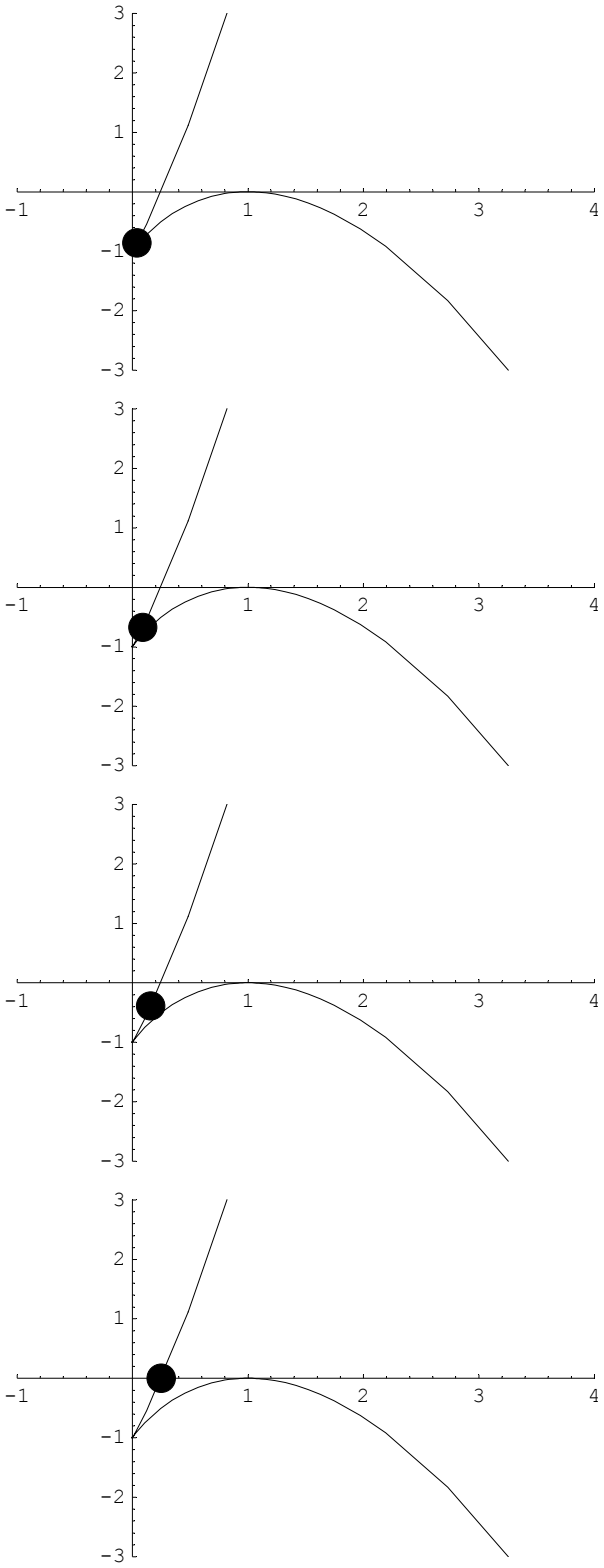

3
2
1
-1
-2
-3
-1
1
2
3
4

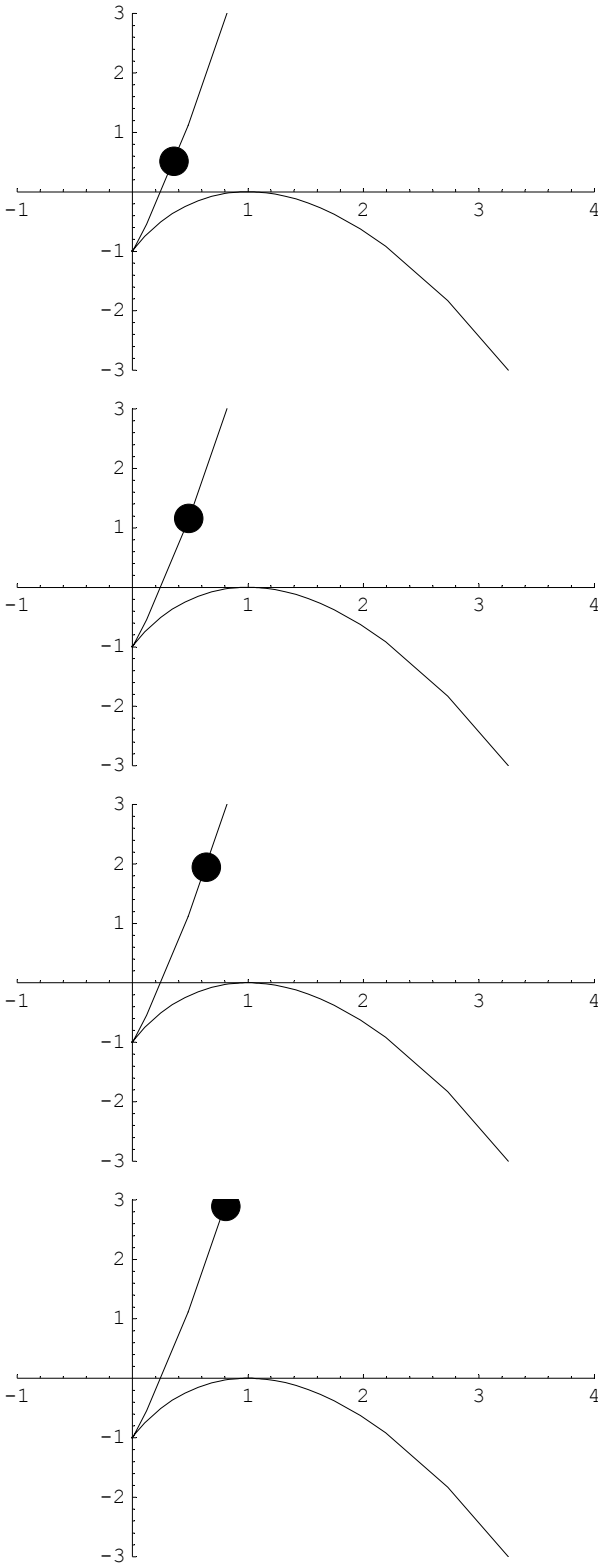

3
2
1
-1
-2
-3
-1
1
2
3
4

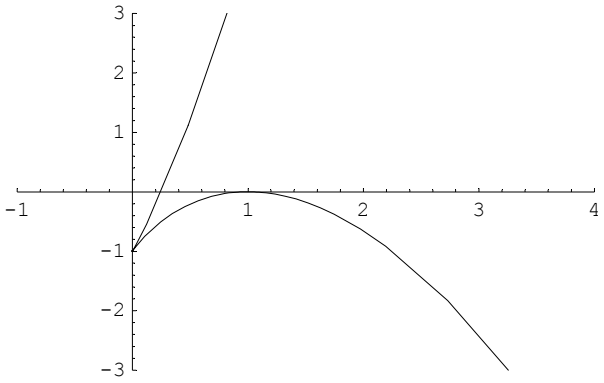

3
2
1
-1
-2
-3
-1
1
2
3
4


3
2
1
-1
1
2
3
4
-1
-2
-3


3
2
1
-1
1
2
3
4
-1
-2
-3


3
2
1
-1
1
2
3
4
-1
-2
-3


3
2
1
-1
-2
-3
-1
1
2
3
4



However, the book did not ask us to understand this motion. They asked us just to calculate the velocity

```
In[34]:= D[{(t - 1)^2, 3 t^2 - 2 t^3}, t]

Out[34]= {2 (-1 + t), 6 t - 6 t^2}
```

And to calculate the magnitude of the velocity vector, that is the speed:

```
In[35]:= √{2 (t - 1), 6 t^2 - 6 t}.{2 (t - 1), 6 t^2 - 6 t}

Out[35]= √(4 (-1 + t)^2 + (-6 t + 6 t^2)^2)

In[36]:= FullSimplify[√((2 (t - 1))^2 + (6 t^2 - 6 t)^2)]

Out[36]= 2 √((-1 + t)^2 (1 + 9 t^2))
```

Which can be simplified to

```
2 Abs[t - 1] √(1 + 9 t^2)
```