MATH 226 Examination 1 April 28, 2009

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Give all details of your reasoning. Each problem is worth 25 points for the total of 100 points.

Problem 1. (a) Write without the absolute values the exact value of the expression

$$|\pi^e - e^\pi| .$$

since $e^\pi \doteq 23.1$ and $\pi^e \doteq 22.5$
$e^\pi > \pi^e$ so $|\pi^e - e^\pi| = e^\pi - \pi^e$

(b) Write the following English sentence as an inequality involving absolute value:

The distance between a number x and the number $-\frac{2}{3}$ is less than $\frac{1}{4}$.

Illustrate with a diagram on the number line.

$\left| x + \frac{2}{3} \right| < \frac{1}{4}$

see scratch for # line

Problem 2. (a) State the definition of the absolute value function.

(b) State all the properties of absolute value that you will need in (c). (No proofs are required, just the statements. You can not list any version of the triangle inequality here.)

(c) Prove that $|a + b| \le |a| + |b|$ for all $a, b \in \mathbb{R}$.

See scratch

Problem 3. (a) State the definition of

$$\lim_{x \to +\infty} f(x) = L .$$

See scratch

(b) Use the definition of limit to prove that

$$\lim_{x \to +\infty} \frac{x}{x + \cos x} = ? .$$

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Problem 4. (a) State the ϵ-δ definition of continuity of a function f at a point a.

(b) Use ϵ-δ definition of continuity to prove that the function

$$f(x) = \frac{1}{x^2}$$

is continuous on $(0, +\infty)$.

See scratch

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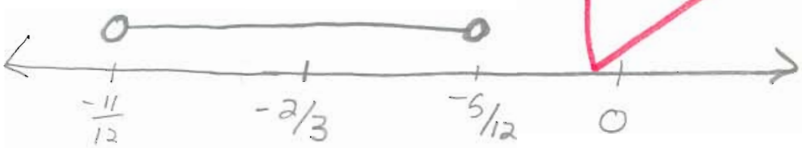
1 b. $\left| x + \frac{2}{3} \right| < \frac{1}{4}$



$x + \frac{2}{3} < \frac{1}{4}$

$x < -5/12$

$x + \frac{2}{3} > -\frac{1}{4}$

$x > -\frac{11}{12}$

This means that we can go less than $\frac{1}{4}$ to the right, or left. If we move to the right $\frac{1}{4}$, we arrive at $-5/12$. If we move to the left $\frac{1}{4}$, we arrive at $-\frac{11}{12}$. So our solns lie on $\left(-\frac{11}{12}, -\frac{5}{12}\right)$

2. a. $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

b. ① $x \leq |x|$

② $-x \leq |x|$

③ $|x| = \max\{x, -x\}$

c. Case I

By ① $a \leq |a|$ $\quad b \leq |b|$

$a + b \leq |a| + |b|$

Case II

By ② $-a \leq |a|$ $\quad -b \leq |b|$

$-(a+b) \leq |a| + |b|$

Since $\max\{a+b, -(a+b)\} \leq |a| + |b|$

$|a+b| \leq |a| + |b|$ by ③

③ $\lim\limits_{x\to\infty} f(x) = L$

I. $\exists\ X_0 \in \mathbb{R}$ s.t. $f(x)$ is defined for all $x \geq X_0$

II. $\forall\ \varepsilon > 0\ \exists\ x(\varepsilon) \in \mathbb{R}$ s.t. $x(\varepsilon) \geq X_0$ and

$$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$$

b. $\lim\limits_{x\to\infty} \dfrac{x}{x+\cos x} = 1$

I. $f(x)$ is defined for $x > 1$ so take $X_0 = 2$

$[2, \infty)$

Prove $\left| \dfrac{x}{x+\cos x} - 1 \right| < \varepsilon$

$$\left| \frac{x}{x+\cos x} - 1 \right| = \left| \frac{x-(x+\cos x)}{x+\cos x} \right| = \left| \frac{-\cos x}{x+\cos x} \right| = \frac{|-\cos x|}{|x+\cos x|}$$

$$\frac{|-\cos x|}{|x+\cos x|} \underset{\substack{\uparrow \\ \text{since} \\ |-a| = |a|}}{=} \frac{|\cos x|}{|x+\cos x|} \underset{\substack{\uparrow \\ x+\cos x > 0 \\ \text{for } x \geq 2}}{=} \frac{|\cos x|}{x+\cos x} \underset{\substack{\uparrow \\ 1 \geq |\cos x|}}{\leq} \frac{1}{x+\cos x} \underset{\substack{\uparrow \\ \frac{1}{2}x \leq x+\cos x}}{\leq} \frac{1}{\frac{1}{2}x} = \frac{2}{x}$$

$\frac{1}{2}x$ is smallest, on our interval @ 2

$\frac{d}{dx}(x+\cos x) = 1 - \sin x$

min @ $n\pi$

$\pi + \cos\pi = \pi - 1 = 2.14$

since $2.14 > 2$ $x + \cos x \geq \frac{1}{2}x$

and $\dfrac{1}{x+\cos x} \leq \dfrac{1}{\frac{1}{2}x}$

$\dfrac{2}{x} < \varepsilon \qquad \dfrac{2}{\varepsilon} < x$

$$x(\varepsilon) = \max\left\{ \frac{2}{\varepsilon}, 2 \right\}$$

If $x > x(\varepsilon)$ then $x > 2$, satisfying (I)

and $x > \dfrac{2}{\varepsilon} \rightarrow \dfrac{2}{x} < \varepsilon$

since $\left| \dfrac{x}{x+\cos x} - 1 \right| \leq \dfrac{2}{x}$ and $\dfrac{2}{x} < \varepsilon$ then $\left| \dfrac{x}{x+\cos x} - 1 \right| < \varepsilon$ satisfying (II)

$$\lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2}$$

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(4) a

I. $\exists$ $\delta_0 > 0$ s.t. $f(x)$ is defined for all x in

$(a - \delta_0, a + \delta_0)$

II $\forall \varepsilon > 0$ $\exists$ $\delta(\varepsilon)$ $0 < \delta(\varepsilon) \leq \delta_0$ s.t. $|x-a| < \delta(\varepsilon)$

$\Rightarrow |f(x) - f(a)| < \varepsilon$

b. $f(x) = \frac{1}{x^2}$  on $(0, \infty)$

I Take $\delta_0 = \frac{a}{2}$

Our world is $\left(\frac{a}{2}, \frac{3a}{2}\right)$ $a > 0$ b/c of interval $(0, \infty)$

II Prove $\left| \frac{1}{x^2} - \frac{1}{a^2} \right| < \varepsilon$

$$\left| \frac{a^2 - x^2}{a^2 x^2} \right| = \frac{|a^2 - x^2|}{|a^2 x^2|} \underset{\uparrow}{=} \frac{|a-x||a+x|}{a^2 x^2} = \frac{|x-a||x+a|}{a^2x^2}$$

by $|a| = |-a|$

$|xy| = |x||y|$

(+) in our world

$a > 0$ so drop abs. value

$$\frac{|x-a||x+a|}{a^2x^2} \leq \frac{|x-a||x+a|}{a^2} \leq \frac{|x-a|\left|\frac{3a}{2} + a\right|}{a^2} = |x-a| \cdot \frac{4a}{2a^2} = \frac{2}{a} < \varepsilon$$

Clearly $a^2x^2 \geq a^2$ so $\frac{1}{a^2x^2} \leq \frac{1}{a^2}$ Not true for all $a > 0$

$x + a \leq \frac{3a}{2}$

?

$|x - a| < \frac{\varepsilon a}{2}$ $\delta(\varepsilon) = \min\left\{ \boxed{\frac{\varepsilon a}{2}}, \frac{a}{2} \right\}$

satisfying I, satisfying II,

$|x-a| \leq \frac{a}{2}$ and $|x-a| < \frac{\varepsilon a}{2}$ By the nature of a minimum

Since $|x-a| < \frac{\varepsilon a}{2}$, then $|x-a| \cdot \frac{\varepsilon a}{2} < \varepsilon$ and since $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| \leq |x-a| \cdot \frac{\varepsilon a}{2}$ so $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon$