MATH 226

Final Examination
June 10, 2009

Name Key

Give all details of your reasoning.

Problem 1. (a) Prove that $|x+y| \leq |x| + |y|$ for all real numbers x and y.

(b) Prove that $\big||a| - |b|\big| \leq |a-b|$ for all real numbers a and b.

Problem 2. (a) State the definition of $\lim_{x\to+\infty} f(x) = L$.

(b) Use the definition of limit to prove that

$$\lim_{x\to+\infty} \frac{x+\sqrt{x}}{x-(\sin x)\sqrt{x}} = 1 \,.$$

Problem 3. (a) State the ϵ-δ definition of continuity of a function f at a point a.

(b) Use the ϵ-δ definition of continuity to prove that the function $f(x) = \dfrac{1}{1+x^2}$ is continuous on its domain.

Problem 4. (a) State the definition (using ϵ) of a convergent sequence.

(b) State the definition of a bounded sequence.

(c) Prove that a convergent sequence is bounded.

Problem 5. (a) Consider the sequence

$$a_1 = 3, \qquad a_{n+1} = \sqrt{1+a_n}, \quad n \in \mathbb{N}.$$

Prove that this sequence converges and find its limit.

(b) State clearly the most important theorem which you used in the proof.

Problem 6. Find the sums of the following three series:

$$\text{(A)} \quad \sum_{n=1}^{+\infty} \frac{3^{n+2}}{2^{2n}}; \qquad \text{(B)} \quad \sum_{n=1}^{+\infty} \frac{1}{n(n+1)}; \qquad \text{(C)} \quad \sum_{n=2}^{+\infty} \frac{2}{n^2-1}.$$

Problem 7. Use convergence tests to decide whether the series is absolutely convergent, conditionally convergent, or divergent. Explain your reasoning: State clearly which test is being used and make sure that all requirements of that test are fulfilled.

$$\text{(A)} \quad \sum_{n=1}^{+\infty} \frac{\cos((n-1)\pi)}{n^2}; \qquad \text{(B)} \quad \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{\sqrt{n}}; \qquad \text{(C)} \quad \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2-(-1)^n}.$$

Problem 8. (a) Find the domain of the function $f(x) = \sum_{k=1}^{+\infty} \frac{1}{k} x^k$. (Pay special attention to the end-points of the interval.) Calculate $f(0)$, $f'(0)$, $f''(0)$.

(b) Calculate the power series for $g(x) = f'(x)$. Here $f(x)$ is given in part (a). What is the domain of g?

(c) Find simple formulas for the functions $g(x)$ and $f(x)$ given in parts (b) and (a).

(d) Based on (a), (b) and (c) calculate the exact values of $\sum_{k=1}^{+\infty} \frac{1}{k\,2^k}$ and $\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$.

$\boxed{1}$

① ⓐ We know that $|x| = \max\{x, -x\}$,
and $|y| = \max\{y, -y\}$.

Therefore $x \le |x|$ and $-x \le |x|$
and $y \le |y|$ and $-y \le |y|$

Thus $x+y \le |x|+|y|$ and $-(x+y) \le |x|+|y|$.

Hence $\max\{x+y, -(x+y)\} \le |x|+|y|$,

that is $|x+y| \le |x|+|y|$.

ⓑ In the last inequality set $x = a-b, y = b$.

Then $|a| \le |a-b| + |b|$.

Hence $|a| - |b| \le |a-b|$. ⊛

In the last inequality swap a and b
$$|b| - |a| \le |b-a| = |a-b|.$$

Thus $-(|a|-|b|) \le |a-b|$ ⊛⊛

⊛ and ⊛⊛ yield
$$\max\{|a|-|b|, -(|a|-|b|)\} \le |a-b|.$$

~~The~~ Consequently $\big||a|-|b|\big| \le |a-b|$.

② ⓐ ~~For every $\varepsilon > 0$ there exists~~

(I) There exists $X_0 \in \mathbb{R}$ such that $f(x)$ is defined for all $x \ge X_0$.

(II) For every $\varepsilon > 0$ there exists $X(\varepsilon) \ge X_0$ such that
$$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon.$$

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(2) (b)(I) set $\boxed{X_0 = 2}$. Then, for $x \geq X_0$ we have

$$1 \geq \sin x$$

and

$$x > \sqrt{x}$$

$\sqrt{x} > 0$

Since $\sqrt{x} > 0$, we have $\sqrt{x} \geq (\sin x)\sqrt{x}$.

Hence $x > \sqrt{x} \geq (\sin x)\sqrt{x}$. Thus

$$x - (\sin x)\sqrt{x} > 0 \quad \forall x \geq X_0.$$

So $f(x) = \dfrac{x+\sqrt{x}}{x-(\sin x)\sqrt{x}}$ is defined for $x \geq 2$.

(II) Let $\varepsilon > 0$. Now solve $x \geq X_0$

$$\left| \frac{x+\sqrt{x}}{x-(\sin x)\sqrt{x}} - 1 \right| < \varepsilon$$

Simplify

$$\left| \frac{x+\sqrt{x}}{x-(\sin x)\sqrt{x}} - 1 \right| = \frac{|x+\sqrt{x}-x+(\sin x)\sqrt{x}|}{x-(\sin x)\sqrt{x}}$$

$$= \frac{|\sqrt{x}\,(1+\sin x)|}{x-(\sin x)\sqrt{x}} = \frac{\cancel{\sqrt{x}}\,(1+\sin x)}{\cancel{\sqrt{x}}\,(\sqrt{x}-\sin x)}$$

bigger pizza

$$\leq \frac{2}{\sqrt{x}-1}$$

smaller party

Now $\dfrac{2}{\sqrt{x}-1} < \varepsilon$ is easy to solve

3

For $x \geq X_0$ $\frac{1}{\sqrt{x}-1} < \frac{\varepsilon}{2} \iff$

$$\sqrt{x} - 1 > \frac{2}{\varepsilon} \iff$$

$$\sqrt{x} > \frac{2}{\varepsilon} + 1 \iff$$

$$x > \left(\frac{2}{\varepsilon}+1\right)^2$$

Set $X(\varepsilon) = \max\left\{2, \left(\frac{2}{\varepsilon}+1\right)^2\right\}$.

Recall that we proved

$$|f(x)-1| < \frac{2}{\sqrt{x}-1} \quad \text{for } x \geq 2$$

and $\frac{2}{\sqrt{x}-1} < \varepsilon \iff x > \left(\frac{2}{\varepsilon}+1\right)^2$ for $x \geq 2$.

Let $x > X(\varepsilon)$. Then $x > 2$, so

$$\boxed{|f(x)-1| < \frac{2}{\sqrt{x}-1}}.$$

Since $x > X(\varepsilon)$ we have $x > \left(\frac{2}{\varepsilon}+1\right)^2$. Thus $\boxed{\frac{2}{\sqrt{x}-1} < \varepsilon}$. The last two boxed inequalities yield $|f(x)-1| < \varepsilon$.

③ ⓐ A function f is continuous at a if the following two statements are satisfied. $\boxed{4}$

(I) There exists $\delta_0 > 0$ such that $f(x)$ is defined for all $x \in (a-\delta_0, a+\delta_0)$.

(II) For every $\varepsilon > 0$ there exists $\delta(\varepsilon)$, $0 < \delta(\varepsilon) \leq \delta_0$ such that

$$|x-a| < \delta(\varepsilon) \Rightarrow |f(x) - f(a)| < \varepsilon.$$

ⓑ Set $\delta_0 = 1$. Let $a \in \mathbb{R}$ be arbitrary and prove that $f(x) = \frac{1}{1+x^2}$ is continuous at a by constructing $\delta(\varepsilon)$. We need to solve

$$\left| \frac{1}{1+x^2} - \frac{1}{1+a^2} \right| < \varepsilon$$ for $|x-a|$ (Here $\varepsilon > 0$.)

Let $x \in (a-1, a+1)$ and simplify

$$\left| \frac{1}{1+x^2} - \frac{1}{1+a^2} \right| = \frac{|a^2-x^2|}{(1+x^2)(1+a^2)} =$$ bigger pizza.

$$= \frac{|a-x||a+x|}{(1+x^2)(1+a^2)} \leq \frac{|x-a|(|a|+|x|)}{1}$$

smaller party

5

I know $x \in (a-1, a+1)$, so

$$a-1 < x < a+1 \le |a|+1$$

$$-a-1 < -x < -a+1 \le |a|+1$$

Hence $\max\{x, -x\} \le |a|+1$

that is $|x| \le |a|+1$.

Thus, for $x \in (a-1, a+1)$

$$\left| \frac{1}{1+x^2} - \frac{1}{1+a^2} \right| < |x-a| \left(2|a|+1\right) \quad (*)$$

Set $\delta(\varepsilon) = \min\left\{1, \frac{\varepsilon}{2|a|+1}\right\}$

We just solved $|x-a|(2|a|+1) < \varepsilon$ for $|x-a|$.

Now $|x-a| < \delta(\varepsilon) \Rightarrow \left| \frac{1}{1+x^2} - \frac{1}{1+a^2} \right| < \varepsilon$
is easy to prove.

$|x-a| < \delta(\varepsilon) \Rightarrow x \in (a-1, a+1)$ and $|x-a| < \frac{\varepsilon}{2|a|+1}$.

Since $x \in (a-1, a+1)$, we have

$$\left| \frac{1}{1+x^2} - \frac{1}{1+a^2} \right| < |x-a|(2|a|+1) < \varepsilon.$$

5 a The given sequence is bounded below by 0: 6

$a_n > 0 \quad \forall n \in \mathbb{N}$.

Clearly $a_1 > 0$. Assume $k \in \mathbb{N}$ and $a_k > 0$. Then $a_{k+1} = \sqrt{1+a_k} > \sqrt{1} = 1$.

Hence $a_{k+1} > 0$.

The given sequence is decreasing:

$a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

Proof is again by mathematical induction.

For $n=1$ $\quad a_1 = 3 > a_2 = 2$.

Assume $k \in \mathbb{N}$ and $a_k > a_{k+1}$. Then

$1 + a_k > 1 + a_{k+1}$. Thus

$\sqrt{1+a_k} > \sqrt{1+a_{k+1}}$, that is

$a_{k+1} > a_{k+2}$.

By Math. Induction this proves that the sequence is decreasing.

b We proved that the sequence is bounded below and decreasing.

By the Monotone Convergence Theorem which says: $\boxed{7}$

$\boxed{\text{If a sequence is bounded and monotonic it converges.}}$

our sequence converges.

Set $\lim_{n\to\infty} a_n = L$ Clearly $\boxed{L>0}$

Since $a_{n+1}^2 = 1 + a_n$

and the algebra of limits we conclude $L^2 = 1 + L$.

Solving we get $L = \frac{1+\sqrt{5}}{2}$.

$\boxed{\text{For Problem 4 see Exam 2.}}$

6 A

8

$$\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{2n}} = \sum_{n=1}^{\infty} \frac{9 \cdot 3^n}{4^n} =$$

$$= \underbrace{9 \cdot \frac{3}{4}}_{a} + \underbrace{9 \cdot \frac{9}{16}}_{ar} + \underbrace{9 \cdot \frac{27}{64}}_{ar^2} + \dots + \underbrace{9 \cdot \frac{3}{4}\left(\frac{3}{4}\right)^n}_{a \cdot r^n} + \dots$$

This is a geometric series with

$$a = \frac{27}{4} \text{ and } r = \frac{3}{4}$$

so $$\sum_{n=1}^{\infty} \frac{3^{n+2}}{2^{2n}} = \frac{a}{1-r} = \frac{\frac{27}{4}}{1-\frac{3}{4}}$$

$$= 27$$

B Consider partial sums:

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} =$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) \cancel{+} \dots \cancel{+} \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Clearly $\lim_{n \to \infty} s_n = 1$. So

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

(c) Consider partial sums

$\boxed{9}$

$$S_n = \frac{2}{2^2-1} + \frac{2}{3^2-1} + \cdots + \frac{2}{n^2-1} =$$

$$= \frac{2}{(2-1)(2+1)} + \frac{2}{(3-1)(3+1)} + \cdots + \frac{2}{(n-1)(n+1)}$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \cdots +$$

$$\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right)$$

$$\left(\cancel{\frac{1}{n-2}} - \frac{1}{n}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)$$

So, $S_n = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$.

Verify: $S_2 = \frac{3}{2} - \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} = \frac{2}{3}$ ✓

$\frac{2}{3} + \frac{2}{8} = \frac{16+6}{24} = S_3 = \frac{3}{2} - \frac{1}{3} - \frac{1}{4} = \frac{5}{4} - \frac{1}{3} = \frac{15-4}{12} = \frac{11}{12}$

$\frac{11}{12} + \frac{2}{15} = \frac{55+8}{60} = S_4 = \frac{3}{2} - \frac{1}{4} - \frac{1}{5} = \frac{5}{4} - \frac{1}{5} = \frac{25-4}{20} = \frac{21}{20}$

$= \frac{63}{60} = \frac{21}{20}$ ✓

$$\lim_{n\to\infty} S_n = \frac{3}{2}$$

7 A $\cos((n-1)\pi) = (-1)^{n+1}$ $\boxed{10}$

So the sequence is alternating.

But $\left|\dfrac{\cos((n-1)\pi)}{n^2}\right| = \dfrac{1}{n^2}$

So this sequence <u>converges absolutely.</u>

B The sequence of absolute values is $\dfrac{1}{\sqrt{n}}$ and the series $\sum \dfrac{1}{\sqrt{n}}$ diverges. ~~So~~ But the series $\sum \dfrac{(-1)^{n+1}}{\sqrt{n}}$ is alternating and it converges by the alt. series test since $\dfrac{1}{\sqrt{n}} > 0$, $\dfrac{1}{\sqrt{n}} > \dfrac{1}{\sqrt{n+1}}$ and $\lim_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$.

C Consider the ~~se~~ series of absolute values $\sum_{n=1}^{\infty} \dfrac{1}{n^2-(-1)^n}$

$$= 1 + \frac{1}{2^2-1} + \frac{1}{3^2+1} + \frac{1}{4^2-1} + \cdots$$

By comparison with the series in 6 c this series converges. We have $\boxed{11}$

$$\frac{1}{n^2-(-1)^n} \leq \frac{1}{n^2-1}, \; n \geq 2.$$

So the sum of the series of absolute values is $\leq 1+\frac{1}{2}\frac{3}{2}=\frac{7}{4}$.
So the series converges absolutely.

8 a $\qquad a_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{|a_{n+1}||x|^{n+1}}{|a_n||x|^n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}|x|}{\frac{1}{n}} = |x|$$

So the domain of the function is $(-1, 1)$. But at $x=-1$ the series is alternating harmonic series which converges. At $=1$ the series diverges.
So the domain is

$$[-1, 1).$$

$$f(0)=0$$
$$f'(0)=1$$
$$f''(0)=1$$

$$f(x)=x+\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots$$ $\boxed{12}$
$$f'(x)=1+x+x^2+\cdots$$
$$f''(x)=1+2x+\cdots$$

8 b $g(x)=1+x+x^2+\cdots=\frac{1}{1-x}$

the domain for $g(x)$ is $(-1,1)$.

8 c $g(x)=\frac{1}{1-x}$

$f'(x)=\frac{1}{1-x}$ $\quad -1<x<1$

$f(x)=\int_0^x \frac{1}{1-t}\,dt=-\ln(1-x)$

8 d $\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{2}\right)^k=f\left(\frac{1}{2}\right)=-\ln\left(1-\frac{1}{2}\right)$

$=\ln 2$

$\sum_{k=1}^{\infty}\frac{(-1)^k}{k}=f(-1)=-\ln(2)=\ln\frac{1}{2}$.