Axioms for the Set $\mathbb{R}$ of Real Numbers

Axiom 1 (A0: Addition is defined)**.** If $a, b \in \mathbb{R}$, then the sum of a and b, denoted by $a + b$, is a uniquely defined number in $\mathbb{R}$.

Axiom 2 (A1: Addition is associative)**.** For all $a, b, c \in \mathbb{R}$ we have $a + (b + c) = (a + b) + c$.

Axiom 3 (A2: Addition is commutative)**.** For all $a, b \in \mathbb{R}$ we have $a + b = b + a$.

Axiom 4 (A3: 0 is a neutral element for addition)**.** There is an element 0 in $\mathbb{R}$ such that $0 + a = a + 0 = a$ for all $a \in \mathbb{R}$.

Axiom 5 (A4: Opposites exist)**.** If $a \in \mathbb{R}$, then the equation $a + x = 0$ has a solution $-a \in \mathbb{R}$. The number $-a$ is called the *opposite* of a.

Axiom 6 (M0: Multiplication is defined)**.** If $a, b \in \mathbb{R}$, then the product of a and b, denoted by ab, is a uniquely defined number in $\mathbb{R}$.

Axiom 7 (M1: Multiplication is associative)**.** For all $a, b, c \in \mathbb{R}$ we have $a(bc) = (ab)c$.

Axiom 8 (M2: Multiplication is commutative)**.** For all $a, b \in \mathbb{R}$ we have $ab = ba$.

Axiom 9 (M3: 1 is a neutral element for multiplication)**.** There is an element $1 \neq 0$ in $\mathbb{R}$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$.

Axiom 10 (M4: Reciprocals exist)**.** If $a \in \mathbb{R}$ is such that $a \neq 0$, then the equation $a \cdot x = 1$ has a solution $a^{-1} = \dfrac{1}{a}$ in $\mathbb{R}$. The number $a^{-1} = \dfrac{1}{a}$ is called the *reciprocal* of a.

Axiom 11 (DL: Distributive law, the connection between addition and multiplication)**.** For all $a, b, c \in \mathbb{R}$ we have $a(b + c) = ab + ac$.

Axiom 12 (O1: Order exists)**.** Given any $a, b \in \mathbb{R}$, exactly one of these statements is true: $a < b$, $a = b$, or $b < a$. (The symbol $a \leq b$ stands for $a < b$ or $a = b$.)

Axiom 13 (O2: Order is transitive)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

Axiom 14 (O3: Order and addition)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ then $a + c < b + c$.

Axiom 15 (O4: Order and multiplication)**.** Given any $a, b, c \in \mathbb{R}$, if $a < b$ and $0 < c$, then $ac < bc$.

Axiom 16 (CA: Completeness Axiom)**.** If A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

All statements about real numbers that are studied in beginning mathematical analysis courses can be deduced from these sixteen axioms.

The formulation of the Completeness Axiom given as Axiom 16 above is not standard. This version I found in the book *Mathematical analysis* by Vladimir Zorich, published by Springer in 2004. The standard formulation of the Completeness Axiom is the boxed statement in the theorem below. In the theorem we prove that Zorich's Completeness Axiom is equivalent to the standard one.

Theorem 1. *The Completeness Axiom (Axiom* 16*) is equivalent to the following statement:*

$$\boxed{\textit{Every nonempty subset of the real numbers that is bounded above has a least upper bound.}}$$

Before giving a proof we notice that the standard Completeness Axiom is given as an English sentence without any formulas. But, to understand it we need definitions of two concepts: "a subset of real numbers is bounded above" and "a least upper bound" of a subset of real numbers. I consider this a drawback, not a feature.

Definition 2. A subset S of real numbers is *bounded above* if there exists a real number M such that $x \leq M$ for all $x \in S$.

A real number v such that $x \leq v$ for all $x \in S$ is called an *upper bound for A*.

A real number u is a *least upper bound for S* if u is an upper bound for S and $u \leq v$ for all upper bounds v of S.

Proof of Theorem 1. Assume that Axiom 16 holds. Let S be a nonempty bounded above subset of $\mathbb{R}$. Let M be an upper bound of S. Let $A = S$ and let B be the set of all upper bounds of A. That is

$$B = \bigl\{ b \in \mathbb{R} : x \leq b \quad \forall x \in S \bigr\}.$$

By assumption $A \neq \emptyset$. Since $M \in B$, also $B \neq \emptyset$. By the definition of B we have

$$a \leq b \qquad \text{for all} \quad a \in A \quad \text{and for all} \quad b \in B.$$

By Axiom 16 we conclude that there exists $c \in \mathbb{R}$ such that

$$a \leq c \leq b \qquad \text{for all} \quad a \in A \quad \text{and for all} \quad b \in B.$$

The last relation has two components:

$$a \leq c \qquad \text{for all} \quad a \in A = S,$$

which means that c is an upper bound for S, and

$$c \leq b \qquad \text{for all} \quad b \in B,$$

which means that $c \leq b$ for all upper bounds b of S. Therefore c is a least upper bound for S.

Now assume that "Every nonempty subset of the real numbers that is bounded above has a least upper bound." We will prove the statement in Axiom 16. Let A and B be nonempty subsets of $\mathbb{R}$ such that

$$a \leq b \qquad \text{for all} \quad a \in A \quad \text{and for all} \quad b \in B. \tag{1}$$

Since B is nonempty there exists $v \in B$. By the inequalities (1) we have $a \leq v$ for all $a \in A$. Thus, the set A is bounded above. Since A is also nonempty by the assumption that "Every nonempty subset of real numbers that is bounded above has a least upper bound" we conclude that A has a least upper bound, call it u. Since u is an upper bound for A we have

$$a \leq u \qquad \text{for all} \quad a \in A. \tag{2}$$

By (1), each element in B is an upper bound for A. Since u is a least upper bound for A we conclude

$$u \leq b \qquad \text{for all} \quad b \in B. \tag{3}$$

The relations (2) and (3) yield

$$a \leq u \leq b \qquad \text{for all} \quad a \in A \quad \text{and for all} \quad b \in B.$$

Thus, we can set $c = u$ and Axiom 16 is proved. $\square$