7 Sequences of real numbers

7.1 Definitions and examples

Definition 7.1.1. A sequence of real numbers is a real function whose domain is the set $\mathbb{N}$ of natural numbers.

Let $s : \mathbb{N} \to \mathbb{R}$ be a sequence. Then the values of s are $s(1), s(2), s(3), \dots, s(n), \dots$. It is customary to write s_n instead of $s(n)$ in this case. Sometimes a sequence will be specified by listing its first few terms

$$s_1,\ s_2,\ s_3,\ s_4,\ \dots\ ,$$

and sometimes by listing of all its terms $\{s_n\}_{n\in\mathbb{N}}$ or $\{s_n\}_{n=1}^{+\infty}$. One way of specifying a sequence is to give a formula, or recursion formula for its $n-$th term s_n. Notice that in this notation s is the "name" of the sequence and n is the variable.

Some examples of sequences follow.

Example 7.1.2. (a) $1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ \dots;$

(b) $1,\ 0,\ 1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 1,\ 1,\ 1,\ 1,\ 0,\ 1,\ \dots;$

(c) $1,\ 1,\ 1,\ 1,\ 1,\ \dots;$ (the constant sequence)

(d) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \dots;$ (What is the range of this sequence?)

Recursively defined sequences

Example 7.1.3. (a) $x_1 = 1, \quad x_{n+1} = 1 + \dfrac{x_n}{4}, \ n = 1, 2, 3, \dots;$

(b) $x_1 = 2, \quad x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}, \ n = 1, 2, 3, \dots;$

(c) $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}, \quad n = 1, 2, 3, \dots;$

(d) $s_1 = 1, \quad s_{n+1} = \sqrt{1 + s_n}, \quad n = 1, 2, 3, \dots;$

(e) $x_1 = \dfrac{9}{10}, \quad x_{n+1} = \dfrac{9 + x_n}{10}, \quad n = 1, 2, 3, \dots.$

(f) $b_1 = \dfrac{1}{2}, \quad b_{n+1} = \dfrac{1}{2\sqrt{1 - b_n^2}}, \quad n = 1, 2, 3, \dots$

(g) $f_1 = 1, \ f_{n+1} = (n+1)\, f_n, \quad n = 1, 2, 3, \dots.$

Some important examples of sequences are listed below.

$$b_n = c, \quad c \in \mathbb{R}. \quad n \in \mathbb{N}, \tag{7.1.1}$$
$$p_n = a^n, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{7.1.2}$$
$$x_n = \left(1 + \frac{1}{n}\right)^n, \quad n \in \mathbb{N}, \tag{7.1.3}$$
$$y_n = \left(1 + \frac{1}{n}\right)^{(n+1)}, \quad n \in \mathbb{N}, \tag{7.1.4}$$
$$z_n = \left(1 + \frac{a}{n}\right)^n, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{7.1.5}$$
$$f_1 = 1, \quad f_{n+1} = f_n \cdot (n+1), \quad n \in \mathbb{N}. \tag{7.1.6}$$

(The standard notation for the terms of the sequence $\{f_n\}_{n=1}^{+\infty}$ is $f_n = n!, \ n \in \mathbb{N}$.)

$$q_n = \frac{a^n}{n!}, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}, \tag{7.1.7}$$
$$t_1 = 2, \ t_{n+1} = t_n + \frac{1}{n!} \quad n \in \mathbb{N}, \tag{7.1.8}$$
$$v_1 = 1 + a, \ v_{n+1} = v_n + \frac{a^n}{n!}, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{7.1.9}$$

Let $\{a_n\}_{n=1}^{+\infty}$ be an arbitrary sequence. An important sequence associated with $\{a_n\}_{n=1}^{+\infty}$ is the following sequence

$$S_1 = a_1, \ S_{n+1} = S_n + a_{n+1}, \quad n \in \mathbb{N}. \tag{7.1.10}$$

7.2 Convergent sequences

Definition 7.2.1. A sequence $\{s_n\}_{n=1}^{+\infty}$ of real numbers *converges* to the real number L if for each $\epsilon > 0$ there exists a number $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad |s_n - L| < \epsilon.$$

If $\{s_n\}_{n=1}^{+\infty}$ converges to L we will write

$$\lim_{n \to +\infty} s_n = L \qquad \text{or} \qquad s_n \to L \ \ (n \to +\infty).$$

The number L is called the *limit* of the sequence $\{s_n\}_{n=1}^{+\infty}$. A sequence that does not converge to a real number is said to *diverge*.

Example 7.2.2. Let r be a real number such that $|r| < 1$. Prove that $\lim_{n \to +\infty} r^n = 0$.

Solution. First note that if $r = 0$, then $r^n = 0$ for all $n \in \mathbb{N}$, so the given sequence is a constant sequence. Therefore it converges. Let $\epsilon > 0$. We need to solve $|r^n - 0| < \epsilon$ for n. First simplify $|r^n - 0| = |r^n| = |r|^n$. Now solve $|r|^n < \epsilon$ by taking ln of both sides of the inequality (note that ln is an increasing function)

$$\ln |r|^n = n \ln |r| < \ln \epsilon.$$

Since $|r| < 1$, we conclude that $\ln|r| < 0$. Therefore the solution is $\quad n > \dfrac{\ln \epsilon}{\ln|r|}$. Thus, with $N(\epsilon) = \dfrac{\ln \epsilon}{\ln|r|}$, the implication

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \Rightarrow \quad |r^n - 0| < \epsilon$$

is valid. $\square$

Example 7.2.3. Prove that $\displaystyle\lim_{n\to+\infty} \frac{n^2 - n - 1}{2n^2 - 1} = \frac{1}{2}$.

Solution. Let $\epsilon > 0$ be given. We need to solve $\left|\dfrac{n^2 - n - 1}{2n^2 - 1} - \dfrac{1}{2}\right| < \epsilon$ for n. First simplify:

$$\left|\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\right| = \left|\frac{2}{2}\,\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\,\frac{2n^2 - 1}{2n^2 - 1}\right| = \left|\frac{-2n - 1}{2\,(2n^2 - 1)}\right| = \frac{2n + 1}{4n^2 - 2}$$

Now invent the BIN:

$$\frac{2n + 1}{4n^2 - 2} \leq \frac{2n + n}{4n^2 - 2n^2} = \frac{3n}{2n^2} = \frac{3}{2n}.$$

Therefore the BIN is:

$$\left|\frac{n^2 - n - 1}{2n^2 - 1} - \frac{1}{2}\right| \leq \frac{3}{2n} \quad \text{valid for} \quad n \in \mathbb{N}.$$

Solving for n is now easy:

$$\frac{3}{2n} < \epsilon. \qquad \text{The solution is} \quad n > \frac{3}{2\epsilon}.$$

Thus, with $N(\epsilon) = \dfrac{3}{2\epsilon}$, the implication

$$n > N(\epsilon) \quad \Rightarrow \quad \left|\frac{n^2 - n - 1}{2n^2 - 1} - 1\right| < \epsilon$$

is valid. Using the BIN, this implication should be easy to prove. $\square$

This procedure is very similar to the procedure for proving limits as x approaches infinity. In fact the following two theorems are true.

Theorem 7.2.4. *Let $x \mapsto f(x)$ be a function which is defined for every $x \geq 1$. Assume that $\displaystyle\lim_{x\to+\infty} f(x) = L$. If the sequence $\{a_n\}_{n=1}^{+\infty}$ is defined by*

$$a_n = f(n), \qquad n = 1, 2, 3, \ldots,$$

then $\displaystyle\lim_{n\to+\infty} a_n = L$.

Theorem 7.2.5. *Let $x \mapsto f(x)$ be a function which is defined for every $x \in [-1,0) \cup (0,1]$. Assume that $\lim\limits_{x \to 0} f(x) = L$. If the sequence $\{a_n\}_{n=1}^{+\infty}$ is defined by*

$$a_n = f(1/n), \qquad n = 1, 2, 3, \ldots,$$

then $\lim\limits_{n \to +\infty} a_n = L$.

The above two theorems are useful for proving limits of sequences which are defined by a formula. For example you can prove the following limits by using these two theorems and what we proved in previous sections.

Exercise 7.2.6. Find the following limits. Provide proofs.

(a) $\lim\limits_{n\to+\infty} \sin\left(\frac{1}{n}\right)$ (b) $\lim\limits_{n\to+\infty} n\sin\left(\frac{1}{n}\right)$ (c) $\lim\limits_{n\to+\infty} \ln\left(1+\frac{1}{n}\right)$

(d) $\lim\limits_{n\to+\infty} n\ln\left(1+\frac{1}{n}\right)$ (e) $\lim\limits_{n\to+\infty} \cos\left(\frac{1}{n}\right)$ (f) $\lim\limits_{n\to+\infty} \frac{1}{n}\cos\left(\frac{1}{n}\right)$

The Algebra of Limits Theorem holds for sequences.

Theorem 7.2.7. *Let $\{a_n\}_{n=1}^{+\infty}$, $\{b_n\}_{n=1}^{+\infty}$ and $\{c_n\}_{n=1}^{+\infty}$, be given sequences. Let K and L be real numbers. Assume that*

(1) $\lim\limits_{x\to+\infty} a_n = K$,

(2) $\lim\limits_{x\to+\infty} b_n = L$.

Then the following statements hold.

(A) *If $c_n = a_n + b_n$, $n \in \mathbb{N}$, then $\lim\limits_{x\to+\infty} c_n = K + L$.*

(B) *If $c_n = a_n b_n$, $n \in \mathbb{N}$, then $\lim\limits_{x\to+\infty} c_n = KL$.*

(C) *If $L \neq 0$ and $c_n = \dfrac{a_n}{b_n}$, $n \in \mathbb{N}$, then $\lim\limits_{x\to+\infty} c_n = \dfrac{K}{L}$.*

Theorem 7.2.8. *Let $\{a_n\}_{n=1}^{+\infty}$ and $\{b_n\}_{n=1}^{+\infty}$ be given sequences. Let K and L be real numbers. Assume that*

(1) $\lim\limits_{x\to+\infty} a_n = K$.

(2) $\lim\limits_{x\to+\infty} b_n = L$.

(3) *There exists a natural number n_0 such that*

$$a_n \leq b_n \quad \text{for all} \quad n \geq n_0.$$

Then $K \leq L$.

Theorem 7.2.9. *Let $\{a_n\}_{n=1}^{+\infty}$, $\{b_n\}_{n=1}^{+\infty}$ and $\{s_n\}_{n=1}^{+\infty}$ be given sequences. Assume the following*

1. *The sequence $\{a_n\}_{n=1}^{+\infty}$ converges to the limit L.*
2. *The sequence $\{b_n\}_{n=1}^{+\infty}$ converges to the limit L.*
3. *There exists a natural number n_0 such that*

$$a_n \leq s_n \leq b_n \quad \textit{for all} \quad n > n_0.$$

Then the sequence $\{s_n\}_{n=1}^{+\infty}$ converges to the limit L.

Prove this theorem.

7.3 Sufficient conditions for convergence

Many limits of sequences cannot be found using theorems from the previous section. For example, the recursively defined sequences (a), (b), (c), (d) and (e) in Example 7.1.3 converge but it cannot be proved using the theorems that we presented so far.

Definition 7.3.1. Let $\{s_n\}_{n=1}^{+\infty}$ be a sequence of real numbers.

1. If a real number M satisfies

$$s_n \leq M \quad \text{for all} \quad n \in \mathbb{N}$$

then M is called an *upper bound* of $\{s_n\}_{n=1}^{+\infty}$ and the sequence $\{s_n\}_{n=1}^{+\infty}$ is said to be *bounded above.*

2. If a real number m satisfies

$$m \leq s_n \quad \text{for all} \quad n \in \mathbb{N},$$

then m is called a *lower bound* of $\{s_n\}_{n=1}^{+\infty}$ and the sequence $\{s_n\}_{n=1}^{+\infty}$ is said to be *bounded below.*

3. The sequence $\{s_n\}_{n=1}^{+\infty}$ is said to be *bounded* if it is bounded above and bounded below.

Theorem 7.3.2. *If a sequence converges, then it is bounded.*

Proof. Assume that a sequence $\{a_n\}_{n=1}^{+\infty}$ converges to L. By Definition 7.2.1 this means that for each $\epsilon > 0$ there exists a number $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \qquad \Rightarrow \qquad |a_n - L| < \epsilon.$$

In particular for $\epsilon = 1 > 0$ there exists a number $N(1)$ such that

$$n \in \mathbb{N}, \quad n > N(1) \qquad \Rightarrow \qquad |a_n - L| < 1.$$

Let n_0 be the largest natural number which is $\leq\ N(1)$. Then $n_0 + 1, n_0 + 2, \ldots$ are all $>\ N(1)$. Therefore

$$|a_n - L| < 1 \qquad \text{for all} \qquad n > n_0.$$

This means that

$$L - 1 < a_n < L + 1 \qquad \text{for all} \quad n > n_0.$$

The numbers $L-1$ and $L+1$ are not lower and upper bounds for the sequence since we do not know how they relate to the first n_0 terms of the sequence. Put

$$\begin{aligned} m &= \min\{a_1, a_2, \ldots, a_{n_0}, L-1\} \\ M &= \max\{a_1, a_2, \ldots, a_{n_0}, L+1\}. \end{aligned}$$

Clearly

$$\begin{aligned} m \leq a_n & \qquad \text{for all} \quad n = 1, 2, \ldots, n_0 \\ m \leq L - 1 < a_n & \qquad \text{for all} \quad n > n_0. \end{aligned}$$

Thus m is a lower bound for the sequence $\{a_n\}_{n=1}^{+\infty}$.

Clearly

$$\begin{aligned} a_n \leq & M \qquad \text{for all} \quad n = 1, 2, \ldots, n_0 \\ a_n < L + 1 \leq & M \qquad \text{for all} \quad n > n_0. \end{aligned}$$

Thus M is an upper bound for the sequence $\{a_n\}_{n=1}^{+\infty}$. □

Is the converse of Theorem 7.3.2 true? The converse is: If a sequence is bounded, then it converges. Clearly a counterexample to the last implication is the sequence $(-1)^n, n \in \mathbb{N}$. This sequence is bounded but it is not convergent.

The next question is whether boundedness and an additional property of a sequence can guarantee convergence. It turns out that such an property is monotonicity defined in the following definition.

Definition 7.3.3. A sequence $\{s_n\}_{n=1}^{+\infty}$ of real numbers is said to be

non-decreasing if $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$,

strictly increasing if $s_n < s_{n+1}$ for all $n \in \mathbb{N}$,

non-increasing if $s_n \geq s_{n+1}$ for all $n \in \mathbb{N}$.

strictly decreasing if $s_n > s_{n+1}$ for all $n \in \mathbb{N}$.

A sequence with either of these four properties is said to be *monotonic.*

The following two theorems give powerful tools for establishing convergence of a sequence.

Theorem 7.3.4. *If $\{s_n\}_{n=1}^{+\infty}$ is non-decreasing and bounded above, then $\{s_n\}_{n=1}^{+\infty}$ converges.*

Theorem 7.3.5. *If $\{s_n\}_{n=1}^{+\infty}$ is non-increasing and bounded below, then $\{s_n\}_{n=1}^{+\infty}$ converges.*

To prove these theorems we have to resort to the most important property of the set of real numbers: the Completeness Axiom.

The Completeness Axiom. If A and B are nonempty subsets of $\mathbb{R}$ such that for every $a \in A$ and for every $b \in B$ we have $a \leq b$, then there exists $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and all $b \in B$.

Proof of Theorem 7.3.4. Assume that $\{s_n\}_{n=1}^{+\infty}$ is a non-decreasing sequence and that it is bounded above. Since $\{s_n\}_{n=1}^{+\infty}$ is non-decreasing we know that

$$s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_{n-1} \leq s_n \leq s_{n+1} \leq \cdots . \tag{7.3.1}$$

Let A be the range of the sequence $\{s_n\}_{n=1}^{+\infty}$. That is $A = \bigl\{s_n : n \in \mathbb{N}\bigr\}$. Clearly $A \neq \emptyset$. Let B be the set of all upper bounds of the sequence $\{s_n\}_{n=1}^{+\infty}$. Since the sequence $\{s_n\}_{n=1}^{+\infty}$ is bounded above, the set B is not empty. Let $b \in B$ be arbitrary. Then b is an upper bound for $\{s_n\}_{n=1}^{+\infty}$. Therefore

$$s_n \leq b \qquad \text{for all} \qquad n \in \mathbb{N}.$$

By the definition of A this means

$$a \leq b \qquad \text{for all} \qquad a \in A.$$

Since $b \in B$ was arbitrary we have

$$a \leq b \quad \text{for all} \quad a \in A \quad \text{and for all} \quad b \in B.$$

By the Completeness Axiom there exists $c \in \mathbb{R}$ such that

$$s_n \leq c \leq b \qquad \text{for all} \quad n \in \mathbb{N} \quad \text{and for all} \quad b \in B. \tag{7.3.2}$$

Thus c is an upper bound for $\{s_n\}_{n=1}^{+\infty}$ and also $c \leq b$ for all upper bounds b of the sequence $\{s_n\}_{n=1}^{+\infty}$. Therefore, for an arbitrary $\epsilon > 0$ the number $c - \epsilon$ (which is $< c$) is not an upper bound of the sequence $\{s_n\}_{n=1}^{+\infty}$. Consequently, there exists a natural number $N(\epsilon)$ such that

$$c - \epsilon < s_{N(\epsilon)}. \tag{7.3.3}$$

Let $n \in \mathbb{N}$ be any natural number which is $> N(\epsilon)$. Then the inequalities (7.3.1) imply that

$$s_{N(\epsilon)} \leq s_n. \tag{7.3.4}$$

By (7.3.2) the number c is an upper bound of $\{s_n\}_{n=1}^{+\infty}$. Hence we have

$$s_n \leq c \qquad \text{for all} \qquad n \in \mathbb{N}. \tag{7.3.5}$$

Putting together the inequalities (7.3.3), (7.3.4) and (7.3.5) we conclude that

$$c - \epsilon < s_n \leq c \quad \text{for all} \quad n \in \mathbb{N} \ \text{ such that } \ n > N(\epsilon). \tag{7.3.6}$$

The relationship (7.3.6) shows that for $n \in \mathbb{N}$ such that $n > N(\epsilon)$ the distance between numbers s_n and c is $< \epsilon$. In other words

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \text{implies} \quad |s_n - c| < \epsilon.$$

This is exactly the implication in Definition 7.2.1. Thus, we proved that

$$\lim_{n \to +\infty} s_n = c. \qquad \square$$

Example 7.3.6. Prove that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln n, \qquad n = 1, 2, 3, \ldots,$$

converges.

Solution. Use the definition of ln as the integral to prove that for $n > 1$

$$t_n > \int_1^n \left(\frac{1}{\operatorname{Floor}(x)} - \frac{1}{x} \right) dx.$$

Deduce that $t_n > 0$.

Represent

$$t_n - t_{n+1} = \bigl(\ln(n+1) - \ln n\bigr) - \frac{1}{n+1}$$

as an area (or a difference of two areas). Conclude that $t_n - t_{n+1} > 0$.

Now use one of the preceding theorems. □

8 Infinite series of real numbers

8.1 Definition and basic examples

The most important application of sequences is the definition of convergence of an infinite series. From the elementary school you have been dealing with addition of numbers. As you know the addition gets harder as you add more and more numbers. For example it would take some time to add

$$S_{100} = 1 + 2 + 3 + 4 + 5 + \cdots + 98 + 99 + 100$$

It gets much easier if you add two of these sums, and pair the numbers in a special way:

$$\begin{aligned} 2\,S_{100} = \quad & 1 + \ \ 2 + \ \ 3 + \ \ 4 + \cdots + 97 + 98 + 99 + 100 \\ & 100 + 99 + 98 + 97 + \cdots + \ \ 4 + \ \ 3 + \ \ 2 + \ \ \ 1 \end{aligned}$$

A straightforward observation that each column on the right adds to 101 and that there are 100 such columns yields that

$$2\,S_{100} = 101 \cdot 100, \qquad \text{that is} \qquad S_{100} = \frac{101 \cdot 100}{2} = 5050.$$

This can be generalized to any natural number n to get the formula

$$S_n = 1 + 2 + 3 + 4 + 5 + \cdots + (n-1) + n = \frac{(n+1)\,n}{2}.$$

This procedure indicates that it would be impossible to find the sum

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

where the last set of $\cdots$ indicates that we continue to add natural numbers.