For most functions f a proof of $\lim_{x \to a} f(x) = L$ based on the definition in the notes should consist from the following steps.

(1) Find δ_0 such that $f(x)$ is defined for all $x \in (a - \delta_0, a) \cup (a, a + \delta_0)$. Justify your choice.

(2) Use algebra to simplify the expression $|f(x) - L|$ with the assumption that $x \in (a - \delta_0, a) \cup (a, a + \delta_0)$. The quantity $|x - a|$ should appear in this simplification.

(3) Use the simplification from (2) to discover a BIN:

$$\boxed{|f(x) - L| \leq b(|x - a|) \quad \text{valid for all} \quad x \in (a - \delta_0, a) \cup (a, a + \delta_0)}.$$

The content of the box above is a BIN.

Here $b(\cdot)$ should be a simple function with the following properties:

(a) $b(|x - a|) > 0$ for all $x \in (a - \delta_0, a) \cup (a, a + \delta_0)$.

(b) $b(|x - a|)$ is tiny for tiny $|x - a|$.

(c) $b(|x - a|) < \epsilon$ is easily solvable for $|x - a|$. The solution should be of the form

$$|x - a| \; < \; \boxed{\text{some expression involving } \epsilon}.$$

Warning: In the above inequality $\boxed{\text{some expression involving } \epsilon}$ must be tiny when ϵ is tiny.

(4) Use the solution of $b(|x - a|) < \epsilon$, that is $\boxed{\text{some expression involving } \epsilon}$ and δ_0 to define

$$\delta(\epsilon) = \min\left\{ \delta_0, \boxed{\text{some expression involving } \epsilon} \right\}.$$

(5) Use the BIN above to **prove** the implication $0 < |x - a| < \delta(\epsilon) \Rightarrow |f(x) - L| < \epsilon$.

Note: The structure of this **proof** is always the same.

(i) First assume that $0 < |x - a| < \delta(\epsilon)$.

(ii) The definition of $\delta(\epsilon)$ yields that

$$\delta(\epsilon) \leq \delta_0 \quad \text{and} \quad \delta(\epsilon) \leq \boxed{\text{some expression involving } \epsilon}.$$

(iii) Based of (5i) and (5ii) we conclude that the following two inequalities are true:

$$0 < |x - a| < \delta_0 \quad \text{and} \quad |x - a| < \boxed{\text{some expression involving } \epsilon}.$$

(iv) From (3) part (c) we know that

$$|x - a| < \boxed{\text{some expression involving } \epsilon} \quad \Rightarrow \quad b(|x - a|) < \epsilon.$$

Therefore (5iii) yields that $b(|x - a|) < \epsilon$ is true.

(v) We also proved the BIN:

$$\boxed{|f(x) - L| \leq b(|x - a|) \quad \text{valid for all} \quad x \in (a - \delta_0, a) \cup (a, a + \delta_0)}$$

We explained in class that the expressions

$$x \in (a - \delta_0, a) \cup (a, a + \delta_0) \quad \text{and} \quad 0 < |x - a| < \delta_0$$

are equivalent. Thus (5iii) yields that the BIN is true.

(vi) Together $|f(x) - L| \leq b(|x - a|)$ and $b(|x - a|) < \epsilon$ yield

$$|f(x) - L| < \epsilon.$$

Thus the implication $0 < |x - a| < \delta(\epsilon) \Rightarrow |f(x) - L| < \epsilon$ is proved.