$$\lim_{x \to +\infty} \tanh(x) = 1$$

Prove it!

<u>Definition</u> Let $L \in \mathbb{R}$, $D \subseteq \mathbb{R}$, $f: D \to \mathbb{R}$

L is a limit of f as x approaches $+\infty$ if the following two conditions are satisfied:

(I) $\exists X_0 \in \mathbb{R}$ such that $[X_0, +\infty) \subseteq D$

(II) $\forall \varepsilon > 0 \ \exists X(\varepsilon) \geq X_0$ such that

$$\boxed{x > X(\varepsilon)} \Rightarrow \boxed{|f(x) - L| < \varepsilon}$$

In our example $\boxed{|\text{th}(x) - 1| < \varepsilon}$

Example $\boxed{\lim_{x\to+\infty} th(x) = 1}$ Recall $\boxed{th(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}}$ $\forall x \in \mathbb{R}$

Thus dom $th = \mathbb{R}$.

(I) So, we can take $X_0 = 0$

(II) is harder. Let $\varepsilon > 0$ be arbitrary.

To find $X(\varepsilon)$ we need to simplify for $x > 0$

$$|th(x) - 1| = \left|\frac{e^x - e^{-x}}{e^x + e^{-x}} - 1\right| = \left|\frac{e^x - e^{-x} - e^x - e^{-x}}{e^x + e^{-x}}\right| =$$

$$= \left|\frac{-2e^{-x}}{e^x + e^{-x}}\right| = \frac{2e^{-x}}{e^x + e^{-x}} \underset{\text{algebra}}{=} \frac{2}{e^{2x} + 1}$$

$e^x + e^{-x} > 0$

$e^{-x} > 0$

Thus $\boxed{\left|\tanh(x)-1\right| = \frac{2}{e^{2x}+1}}$ True for $\forall x \in \mathbb{R}$ But we need only $x \geq 0$.

Now we need to solve for $x \geq 0$ $\boxed{\frac{2}{e^{2x}+1} < \varepsilon.}$

still too complicated

use Pizza-Party to simplify

$$\frac{2}{e^{2x}+1} < \frac{2}{e^{2x}}$$

Now solve $\boxed{\frac{2}{e^{2x}} < \varepsilon}$ easier to solve true for all $x\in\mathbb{R}$

Solve for x

$$\frac{2}{e^{2x}} < \varepsilon \Leftrightarrow \frac{e^{2x}}{2} > \frac{1}{\varepsilon} \Leftrightarrow e^{2x} > \frac{2}{\underbrace{\varepsilon}_{>0}}$$

$$e^{2x} > \frac{2}{\varepsilon} \Leftrightarrow 2x > \ln\left(\frac{2}{\varepsilon}\right) \Leftrightarrow x > \frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right)$$

Summarize

G1 $\boxed{|\mathrm{th}(x) - 1| < \frac{2}{e^{2x}} \quad \forall x\in\mathbb{R}}$

G2 $\boxed{\frac{2}{e^{2x}} < \varepsilon \Leftrightarrow x > \frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right)}$

Set $X(\varepsilon) = \max\left\{\frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right), 0\right\}$

all green

Now prove: $\boxed{x > \max\left\{\frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right), 0\right\}} \Rightarrow \boxed{|\operatorname{th}(x) - 1| < \varepsilon}$

assume

prove

Assume $\underbrace{x > \max\left\{\frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right), 0\right\}}_{\text{green}} \Rightarrow \underbrace{x > \frac{1}{2}\ln\left(\frac{2}{\varepsilon}\right)}_{\text{green}} \overset{G2}{\Rightarrow} \underbrace{\boxed{\frac{2}{e^{2x}} < \varepsilon}}_{\text{green}}$

By G1 $\underbrace{\boxed{|\operatorname{th}(x) - 1| < \frac{2}{e^{2x}}}}_{\text{green}}$

By transitivity the last two green boxes give $\boxed{|\operatorname{th}(x) - 1| < \varepsilon}$

red is greenified