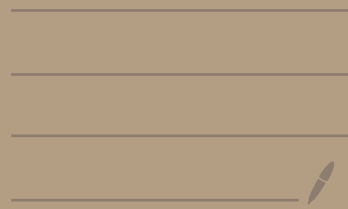


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$$\lim_{x \rightarrow +\infty} \tanh(x) = 1$$

Prove it!



Definition Let  $L \in \mathbb{R}$ ,  $D \subseteq \mathbb{R}$ ,  $f: D \rightarrow \mathbb{R}$

$L$  is a limit of  $f$  as  $x$  approaches  $+\infty$  if the following two conditions are satisfied:

(I)  $\exists X_0 \in \mathbb{R}$  such that  $[X_0, +\infty) \subseteq D$

(II)  $\forall \varepsilon > 0 \exists X(\varepsilon) \geq X_0$  such that

$$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$$

In our example

$$|f(x) - 1| < \varepsilon$$

Example  $\lim_{x \rightarrow +\infty} \text{th}(x) = 1$

Recall  $\text{th}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$   
 $\forall x \in \mathbb{R}$

Thus  $\text{dom th} = \mathbb{R}$ .

(I) So, we can take  $x_0 = 0$

(II) is harder. Let  $\varepsilon > 0$  be arbitrary.

To find  $X(\varepsilon)$  we need to simplify for  $x > 0$

$$\begin{aligned} | \text{th}(x) - 1 | &= \left| \frac{e^x - e^{-x}}{e^x + e^{-x}} - 1 \right| = \left| \frac{e^x - e^{-x} - e^x - e^{-x}}{e^x + e^{-x}} \right| = \\ &= \left| \frac{-2e^{-x}}{e^x + e^{-x}} \right| \end{aligned}$$

$\uparrow$   $e^x + e^{-x} > 0$   
 $e^{-x} > 0$

$$\frac{2e^{-x}}{e^x + e^{-x}} \stackrel{\text{algebra}}{=} \frac{2}{e^{2x} + 1}$$

Thus  $|f(x) - 1| = \frac{2}{e^{2x} + 1}$  True for  $\forall x \in \mathbb{R}$   
But we need only  $x \geq 0$ .

Now we need to solve for  $x \geq 0$   $\left| \frac{2}{e^{2x} + 1} < \epsilon \right|$

$$\frac{2}{e^{2x} + 1} < \frac{2}{e^{2x}}$$

Now solve

$$\frac{2}{e^{2x}} < \epsilon$$

still too complicated  
use Pizza-Party  
to simplify

easier to solve

true for all  $x \in \mathbb{R}$

Solve for  $x$

$$\frac{2}{e^{2x}} < \varepsilon \Leftrightarrow \frac{e^{2x}}{2} > \frac{1}{\varepsilon} \Leftrightarrow e^{2x} > \frac{2}{\varepsilon} > 0$$

$$e^{2x} > \frac{2}{\varepsilon} \Leftrightarrow 2x > \ln\left(\frac{2}{\varepsilon}\right) \Leftrightarrow x > \frac{1}{2} \ln\left(\frac{2}{\varepsilon}\right)$$

Summarize  $G1$   $|f(x) - 1| < \frac{2}{e^{2x}} \quad \forall x \in \mathbb{R}$

$G2$   $\frac{2}{e^{2x}} < \varepsilon \Leftrightarrow x > \frac{1}{2} \ln\left(\frac{2}{\varepsilon}\right)$

Set  $\delta(\varepsilon) = \max\left\{\frac{1}{2} \ln\left(\frac{2}{\varepsilon}\right), 0\right\}$

all green

Now prove:

$$x > \max \left\{ \frac{1}{2} \ln \left( \frac{2}{\epsilon} \right), 0 \right\} \Rightarrow |th(x) - 1| < \epsilon$$

assume

prove

Assume

$$x > \max \left\{ \frac{1}{2} \ln \left( \frac{2}{\epsilon} \right), 0 \right\} \Rightarrow x > \frac{1}{2} \ln \left( \frac{2}{\epsilon} \right) \Rightarrow \frac{2}{e^{2x}} < \epsilon$$

green                      green                      G2                      green

By G1

$$|th(x) - 1| < \frac{2}{e^{2x}}$$

green

By transitivity the last two green boxes give

$$|th(x) - 1| < \epsilon$$

red is greenified