Two examples of limits with a general a

Proofs based on the definition

for $\lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2}$ for $a > 0$.

and $\lim_{x \to a} \sqrt{x} = \sqrt{a}$ for $a > 0$

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Def. $\lim_{x \to a} f(x) = L$ here $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, $a \in \mathbb{R}$, $L \in \mathbb{R}$

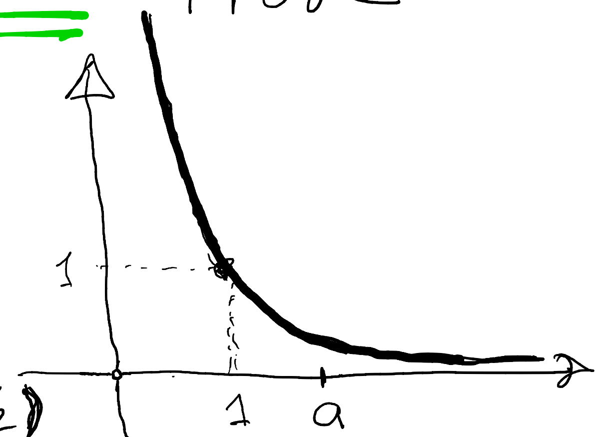
(I) $\exists\, \delta_0 > 0$ s.t. $(a-\delta_0, a) \cup (a, a+\delta_0) \subseteq D$
(f is defined in a neighbourhood of a)

(II) $\forall\, \varepsilon > 0$ $\exists\, \delta(\varepsilon) > 0$ s.t. $\delta(\varepsilon) \le \delta_0$ and
$0 < |x-a| < \delta(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$

Example Let $a \in \mathbb{R}$ s.t. $a \neq 0$. Prove
$$\lim_{x \to a} \frac{1}{x^2} = \frac{1}{a^2}$$

Proof Assume $\boxed{a > 0}$.

(I) $\delta_0 = a/2 > 0$.
since $1/x^2$ is defined on $(a/2, 3a/2)$



(II) Let $\varepsilon > 0$ be arbitrary. Find $\delta(\varepsilon) = ? \leq a/2$

only green allowed

As always we need to solve

$$\boxed{\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon \text{ for } |x-a|} \text{ knowing } \boxed{|x-a| < a/2 \;\; \delta_0} \Leftrightarrow \boxed{x \in \left(\tfrac{a}{2}, \tfrac{3a}{2}\right)}$$

where is ?

Find it!

recall $a > 0$

> 0

do algebra

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \left|\frac{a^2 - x^2}{x^2 a^2}\right| = \frac{|a-x|\,|a+x|}{|x^2 a^2|} = \frac{|x-a|(a+x)}{x^2 a^2} =$$

$x > 0$

here we found $|x-a|$ in $|f(x) - L|$

$$= |\cancel{x} - a|\,\frac{a + \cancel{x}}{\cancel{x^2}a^2} \leq |x-a|\,\frac{5a/2}{\frac{a^2}{4}\cdot a^2} = \frac{10}{a^3}\,|x-a|$$

Pizza - Party

BIN is: $\boxed{\left|\frac{1}{x^2}-\frac{1}{a^2}\right| \leq \frac{10}{a^3}|x-a| \text{ for all } x \in \left(\frac{a}{2}, \frac{3a}{2}\right)}$

To get $\delta(\varepsilon)$ we solve $\frac{10}{a^3}|x-a| < \varepsilon$ for $|x-a|$

The solution is $|x-a| < \frac{a^3}{10}\varepsilon$.

Set $\delta(\varepsilon) = \min\left\{\frac{a^3}{10}\varepsilon, \frac{a}{2}\right\} > 0$

red = green

smells like a solution

To complete the proof. Prove

$$0 < |x-a| < \min\left\{\frac{a^3}{10}\varepsilon, \frac{a}{2}\right\} \Rightarrow \left|\frac{1}{x^2}-\frac{1}{a^2}\right| < \varepsilon$$

Example $\lim_{x \to a} \sqrt{x} = \sqrt{a}$. Here $a > 0$.

(I) $\delta_0 = a/2 > 0$. Clearly $\sqrt{x}$ is defined for all $x \in \left(\frac{a}{2}, \frac{3a}{2}\right)$.

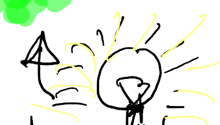
(II) Let $\varepsilon > 0$ be arbitrary. Find $\delta(\varepsilon)$!

$|\sqrt{x} - \sqrt{a}| < \varepsilon$ solve for $|x-a|$ knowing $|x-a| < a/2$

where is it?

Simplify!

$$|\sqrt{x} - \sqrt{a}| = \left| (\sqrt{x} - \sqrt{a}) \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right| = \left| \frac{x-a}{\sqrt{x} + \sqrt{a}} \right|$$



$$= |x-a| \cdot \frac{1}{\sqrt{x} + \sqrt{a}} \le |x-a| \frac{1}{\sqrt{a}}$$

remember Jacob's question!

Pizza Party

BIN: $\boxed{\left|\sqrt{x}-\sqrt{a}\right| \le \frac{1}{\sqrt{a}}|x-a| \text{ for all } x\in\left(\frac{a}{2},\frac{3a}{2}\right)}$

Solve for $|x-a|$: $\frac{1}{\sqrt{a}}|x-a|<\varepsilon$

$$|x-a|<\sqrt{a}\,\varepsilon$$

Set $\delta(\varepsilon)=\min\left\{\sqrt{a}\,\varepsilon,\ \frac{a}{2}\right\}$

Prove: $\boxed{0<|x-a|<\min\left\{\sqrt{a}\,\varepsilon,\ \frac{a}{2}\right\}} \Rightarrow \boxed{\left|\sqrt{x}-\sqrt{a}\right|<\varepsilon}$ R1

Assume

$\boxed{0<|x-a|<\min\left\{\sqrt{a}\,\varepsilon,\ \frac{a}{2}\right\}} \Rightarrow \begin{cases} \boxed{|x-a|<\sqrt{a}\,\varepsilon} \Rightarrow \boxed{\frac{1}{\sqrt{a}}|x-a|<\varepsilon} \text{ G1} \\ \text{and} \\ \boxed{|x-a|<\frac{a}{2}} \Rightarrow \boxed{x\in\left(\frac{a}{2},\frac{3a}{2}\right)} \end{cases}$

The begining of the proof.

Since $\boxed{x \in \left(\frac{a}{2}, \frac{3a}{2}\right)}$ we know that the BIN holds:

$$\boxed{\left|\sqrt{x}-\sqrt{a}\right| \leq \frac{1}{\sqrt{a}}|x-a|}$$

The transitivity of inequality and the BIN and G1 imply

$$\boxed{\left|\sqrt{x}-\sqrt{a}\right| < \varepsilon}$$

the end of the proof

Our goal was to prove R1. Now we have greenified R1, that is we proved it.