A proof of the Sandwich Squeeze Theorem

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a very colorful proof!

and dramatic too: friends foes

Who will win?

Sandwich Squeeze Theorem

f, g, h

Assume:

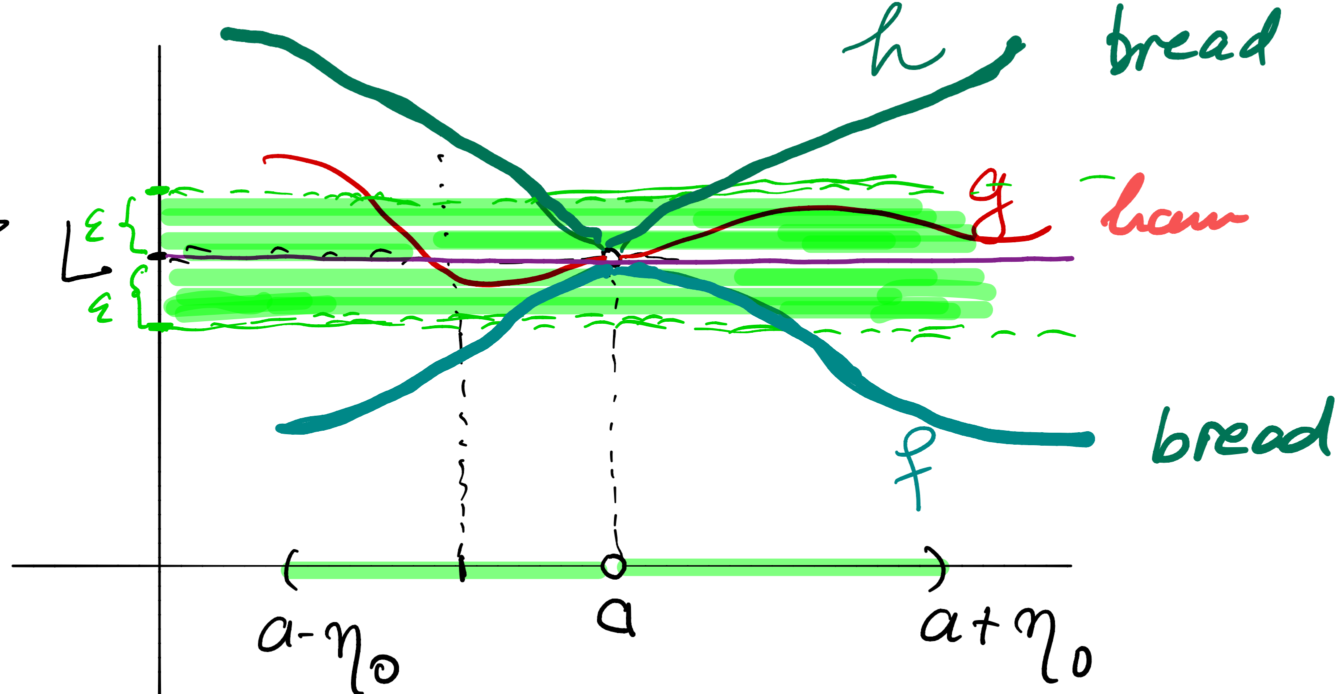
(A) $\lim_{x \to a} f(x) = L$ $\Rightarrow$ (I) f ...
(II) f $\forall \varepsilon > 0\ \exists \delta_f(\varepsilon) > 0$ s.t. $0 < |x-a| < \delta_f(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$

(B) $\lim_{x \to a} h(x) = L$ (I) h ...
(II) h $\forall \varepsilon > 0\ \exists \delta_h(\varepsilon) > 0$ s.t. $0 < |x-a| < \delta_h(\varepsilon) \Rightarrow |h(x) - L| < \varepsilon$

(C) $\exists \eta_0 > 0$ such that $f(x), g(x), h(x)$ defined $\forall x \in (a-\eta_0, a) \cup (a, a+\eta_0)$ and
$\forall x \in (a-\eta_0, a) \cup (a, a+\eta_0)$ $\boxed{f(x) \leq g(x) \leq h(x)}$

Conclusion: $\boxed{\lim_{x \to a} g(x) = L}$ $\Rightarrow$ (I) g ---
(II) g

(X) $\forall \varepsilon > 0\ \exists \delta_g(\varepsilon) > 0$ s.t. $0 < |x-a| < \delta_g(\varepsilon) \Rightarrow |g(x) - L| < \varepsilon$



Let $\varepsilon > 0$ be arbitrary.

How far is $g(x)$ from L?

$|g(x) - L|$

We know that

$$f(x) \le g(x) \le h(x)$$

Here we use
$-|z| \le z \le |z|$

$$\underbrace{-|f(x)-L|}_{\text{friendly quantity}} \le f(x)-L \le g(x)-L \le h(x)-L \le \underbrace{|h(x)-L|}_{\text{friendly quantity}}$$

I CAN ACHIEVE that both $\boxed{|f(x)-L| < \varepsilon \text{ and } |h(x)-L| < \varepsilon}$

$-\varepsilon <$ $\qquad$ $< \varepsilon$

$$-\varepsilon < g(x)-L| < \varepsilon$$

How to make sure that $\boxed{|f(x)-L|<\varepsilon}$?

Just take $\boxed{0<|x-a|<\delta_f(\varepsilon)}$

How to make sure that $\boxed{|h(x)-L|<\varepsilon}$?

Just take $\boxed{0<|x-a|<\delta_h(\varepsilon)}$.

How to have both?

Just take $0<|x-a|<\min\{\delta_f(\varepsilon),\delta_h(\varepsilon)\}$.

This tells me that I can take

$$\delta_g(\varepsilon)=\min\{\delta_f(\varepsilon),\delta_h(\varepsilon),\eta_0\}$$

red = green ☺

Now prove: $\boxed{0<|x-a|<\min\{\delta_f(\varepsilon), \delta_h(\varepsilon), \eta_0\}} \Longrightarrow \boxed{|g(x)-L|<\varepsilon}$

Proof. Assume $\boxed{}$. Then

$\boxed{0<|x-a|<\delta_f(\varepsilon)}$ and $\boxed{0<|x-a|<\delta_h(\varepsilon)}$ and $\boxed{x\in(a-\eta_0,a)\cup(a,a+\eta_0)}$

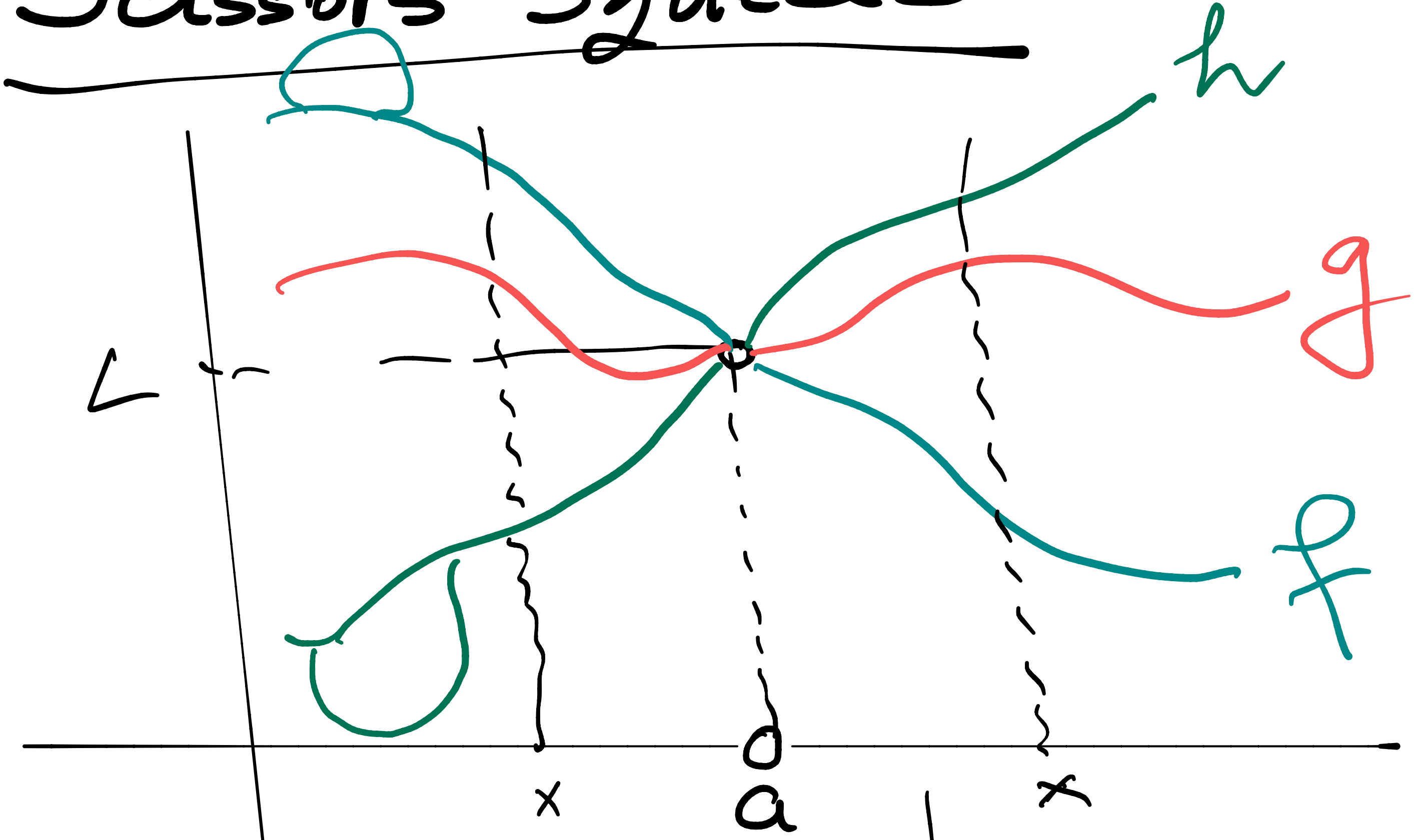
Assump. $\Downarrow$ $\quad\Downarrow\quad\Downarrow$

$\boxed{|f(x)-L|<\varepsilon}$, and $\boxed{|h(x)-L|<\varepsilon}$ and $\boxed{f(x)\le g(x)\le h(x)}$

$$\boxed{-\varepsilon<-|f(x)-L|\le f(x)-L\le g(x)-L\le h(x)-L\le|h(x)-L|<\varepsilon}$$

Red is greenified $\boxed{-\varepsilon<g(x)-L<\varepsilon} \Rightarrow \boxed{|g(x)-L|<\varepsilon}$

Scissors Squeeze

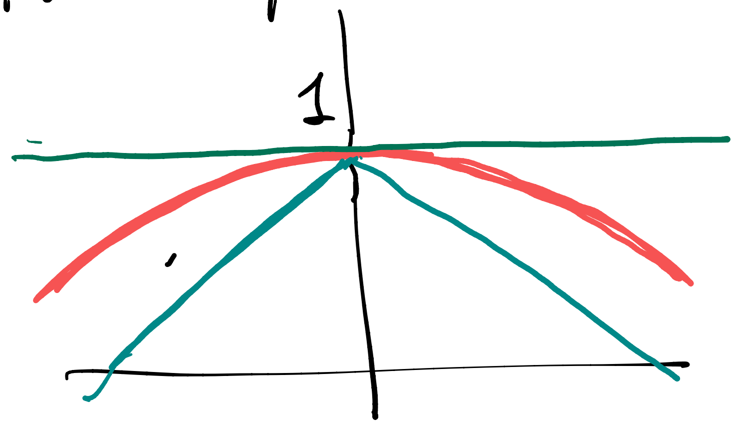


$\forall x \in (a-\eta_0, a)\ h(x) \leq g(x) \leq f(x)$ | $\forall x \in (a, a+\eta_0)\ f(x) \leq g(x) \leq h(x)$

Power of the Squeeze Thm

$$\lim_{x \to 0} \cos x = 1$$ need

How to prove this? We use the definition of $\cos x$ to invent a squeeze



Prove

$$\boxed{1-|x| \le \cos x \le 1}$$

all $x \in (0, \pi/3)$

By the def.

$\cos u \leq 1$

$0 < u \leq \pi/3$