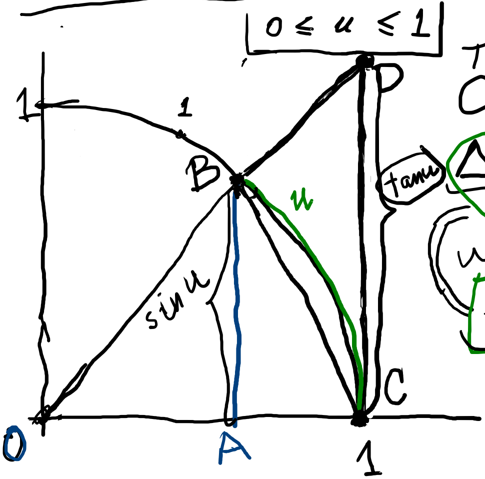
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$ Prove it!

(from yesterday $\forall x \in (-1,1)$
$1-|x| \le \cos x \le 1$
we used geom. arg. to prove)



The core of my argument is :
Compare the areas : circular sector - pizza slice

$$\underbrace{\triangle OCB}_{\text{area}} \le \underbrace{\triangle OCB}_{\text{area}} \le \underbrace{\triangle OCD}_{\text{area}}$$

we consider this visually evident!

$$\frac{1}{2}\sin u \le \frac{1}{2} u \le \frac{1}{2}\frac{\sin u}{\cos u}$$

$$\frac{\sin u}{u} \le 1 \qquad \cos u \le \frac{\sin u}{u}$$

$$\cos u \le \frac{\sin u}{u} \le 1 \quad \text{all } u \in (0,1)$$

$$\boxed{\forall u \in (0,1) \quad \cos u \leq \frac{\sin u}{u} \leq 1}$$

$x \in (-1,0)$, $-x \in (0,1)$, $u = -x$

$$\boxed{\cos(-x) \leq \frac{\sin(-x)}{-x} \leq 1}$$

$$\cos(-x) = \cos x, \qquad \frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin x}{x}$$

$$\boxed{\cos x \leq \frac{\sin x}{x} \leq 1}$$

$\forall x \in (-1,0) \cup (0,1) \Rightarrow$ TRUE

recall

$1 - |x| \leq \cos x \leq 1$

$\Rightarrow$ transitivity

$$\boxed{1 - |x| \leq \frac{\sin x}{x} \leq 1}$$

for all $x \in (-1,0) \cup (0,1)$

Celebrate: We proved $\forall x \in (-1,0) \cup (0,1)$ we have

$$\boxed{1 - |x| \le \frac{\sin x}{x} \le 1} \Rightarrow \boxed{\left|\frac{\sin x}{x} - 1\right| \le |x|}$$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$ by definition:

(I) $\delta_0 = 1$

(II) $\forall \varepsilon > 0 \ \exists \ \delta(\varepsilon) > 0$ s.t. $\delta(\varepsilon) \le \delta_0$ and

$$0 < |x - 0| < \delta(\varepsilon) \Rightarrow \left|\frac{\sin x}{x} - 1\right| < \varepsilon$$

$$\delta(\varepsilon) = \min\{\varepsilon, 1\}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

L'Hospital Rule

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

CIRCULAL REASONING !!

$$(\sin x)' = \cos x$$

$$\frac{d}{dx}(\sin x)\Big|_{x=0} = 1 \quad \lim_{x\to 0}\frac{\sin x - \sin 0}{x-0}$$