A geometric proof for

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$ (based on the Sandwich Squeeze Thm)

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We proved (based on geom. definition of trig. functions)

$$\lim_{x \to 0} \cos x = 1$$

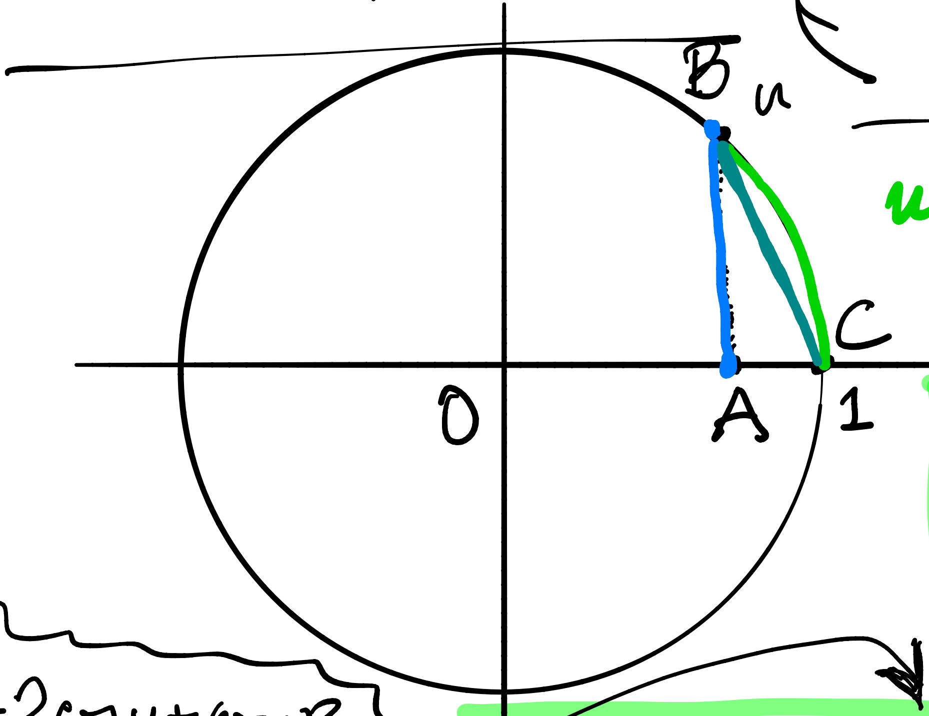
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

The third remarkable trig. limit is

$$\boxed{\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}}$$

Prove this! (by Sandwitch squeeze)

$u \in (0, 1)$



This is clear geometric ineq:

$\overline{AB} \leq \overline{BC} \leq \overset{\frown}{BC}$

PT

straight line shortest dist.

$1 - 2\cos u + (\cos u)^2 + (\sin u)^2 = 2 - 2\cos u$

$$\sin u \leq \sqrt{(1-\cos u)^2 + (\sin u)^2} \leq u$$

$$\boxed{0 < \sin u \le \sqrt{2(1-\cos u)} \le u} \quad \boxed{\forall u \in (0,1)}$$

algebra

$$(\sin u)^2 \le 2(1-\cos u) \le u^2 \quad \Big/ \div 2u^2$$

$$\boxed{\frac{1}{2}\left(\frac{\sin u}{u}\right)^2 \le \frac{1-\cos u}{u^2} \le \frac{1}{2}} \quad \boxed{\forall u \in (0,1)}$$

$$\boxed{\frac{1}{2}\left(\frac{-\sin x}{-x}\right)^2 \le \frac{1-\cos x}{x^2} \le \frac{1}{2}} \quad \boxed{x \in (-1,0)}$$

$-x = u \in (0,1)$

$$\frac{1}{2}\left(\frac{\sin x}{x}\right)^2 \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2} \quad \forall x \in (-1,0)\cup(0,1)$$

Recall $\forall x \in (-1,0)\cup(0,1)$

$$1-|x| \leq \frac{\sin x}{x} \leq 1$$

Thus

$$\frac{1}{2}\left(1-|x|\right)^2 \leq \frac{1-\cos x}{x^2} \leq \frac{1}{2} \quad \forall x \in (-1,0)\cup(0,1)$$

Sandwich Squeeze

Now, to use the Sandwich Squeeze Thm we have to prove:

$$\boxed{\lim_{x \to 0} \frac{1}{2}\left(1-|x|\right)^2 = \frac{1}{2}}$$

and

$$\boxed{\lim_{x \to 0} \frac{1}{2} = \frac{1}{2}}$$

Prove this limit by definition as an exercise!

Prove this limit by def. as an EASY exercise!