Improved definition of continuity.

Example: $f(x) = \frac{1}{x^2+1}, x \in \mathbb{R}$

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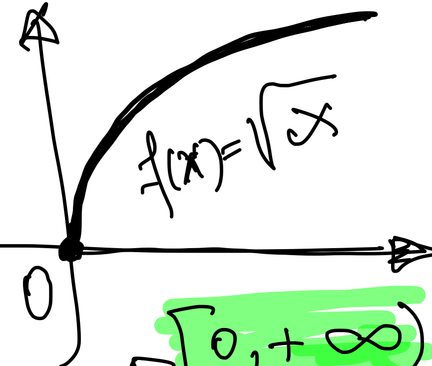
Continuity of functions

$f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$

$n!$ $n \in \{1, 2, 3, \dots\}$



D is an interval

finite interval $a, b \in \mathbb{R}$, $a < b$

(a, b) , $(a, b]$, $[a, b)$, $[a, b]$

$[0, +\infty)$

$(0, +\infty)$

infinite intervals $a \in \mathbb{R}$

$(-\infty, a)$, $(-\infty, a]$, $(a, +\infty)$, $[a, +\infty)$

$\mathbb{R} = (-\infty, +\infty)$

better notation

$-\infty$ $+\infty$ are just symbols not real numbers

Def. A function $f : D \to \mathbb{R}$ is continuous on D if the following condition is satisfied

$$\forall a \in D \ \forall \varepsilon > 0 \ \exists \delta(\varepsilon, a) > 0 \text{ s.t. } \forall x \in D \quad |x - a| < \delta(\varepsilon, a) \Rightarrow |f(x) - f(a)| < \varepsilon$$

Example $f(x)=\sqrt{x}$, $D=[0,+\infty)$.

Proof. Let $a\in D$ be arbitrary.

Consider two cases: **Case 1** $a>0$. Done Tuesday

Case 2. $a=0$.

Let $\varepsilon>0$ be arbitrary. We need to find $\delta(\varepsilon)>0$ s.t. $\forall x\geq 0$ $|x-0|<\delta(\varepsilon)\Rightarrow|\sqrt{x}-\sqrt{0}|<\varepsilon$

$\forall x\geq 0$ $x<\delta\Rightarrow\sqrt{x}<\varepsilon$

Can you solve THIS?

$\delta(\varepsilon)=\varepsilon^2$ Red is green

~~clearly~~ $x\geq 0$

$|x-0|<\varepsilon^2\Rightarrow|\sqrt{x}-\sqrt{0}|<\varepsilon$

simple ALGEBRA

Example $f(x)=\dfrac{1}{x^2+1}$. $D=\mathbb{R}$. Prove f is continuous

Proof. Let $a\in\mathbb{R}$ be arbitrary. Let $\varepsilon>0$ be arbitrary

We need to construct $\delta(\varepsilon, a) > 0$ s.t.

$$\forall x \in \mathbb{R} \quad |x-a| < \delta(\varepsilon, a) \Longrightarrow \left| \frac{1}{x^2+1} - \frac{1}{a^2+1} \right| < \varepsilon$$

The drill is SIMPLIFY:

$$\left| \frac{1}{x^2+1} - \frac{1}{a^2+1} \right| = \left| \frac{a^2+1-(x^2+1)}{(x^2+1)(a^2+1)} \right| =$$

$$= \left| \frac{a^2-x^2}{(x^2+1)(a^2+1)} \right| = \frac{|x^2-a^2|}{|x^2+1|\,|a^2+1|} = \frac{|x-a|\,|x+a|}{(x^2+1)(a^2+1)}$$

Sounds like a math joke: Solve for $|x-a|$ HA, HA

Pizza-Party

$$\leq |x-a| \frac{1+2|a|}{a^2+1}$$

??? ?

P1 in A2

only a allowed

Let us recapitulate what we found so far.

$\forall a \in \mathbb{R}\ \forall x \in \mathbb{R}$ we have $\boxed{\left|\frac{1}{x^2+1} - \frac{1}{a^2+1}\right| = |x-a| \frac{|x+a|}{(x^2+1)(a^2+1)}}$. **G1**

Recall that we need to solve $\underbrace{|x-a| \bigcirc}_{\text{The problem with this expression}} < \varepsilon$ for $|x-a|$.

Now we need to replace $\bigcirc$ with something bigger independent of x. One can proceed in many different ways, depending on your inspiration.

The problem with this expression is that $\bigcirc$ involves x, so we cannot divide by $\bigcirc$, that would not constitute a solution for $|x-a|$ since the right-hand side would involve x. (which is red in this setting)

I claim:

$\forall a \in \mathbb{R}\ \forall x \in \mathbb{R}$ we have $\boxed{\frac{|x+a|}{(x^2+1)(a^2+1)} \leq 1}$. **DC**

This is of course a daring claim. Here is a proof.

Let $a \in \mathbb{R}$ and $x \in \mathbb{R}$ be arbitrary

$$\boxed{\frac{|x+a|}{(x^2+1)(a^2+1)} \leq} \frac{|x|+|a|}{(x^2+1)(a^2+1)} = \frac{|x|}{(x^2+1)(a^2+1)} + \frac{|a|}{(x^2+1)(a^2+1)} \boxed{\leq \frac{|x|}{x^2+1} + \frac{|a|}{a^2+1}}$$ G2

Pizza-Party and Triangle Ineq.

ALGEBRA

Pizza-Party

Now make a short intermezzo to prove that

$$\forall x \in \mathbb{R} \quad \boxed{\frac{|x|}{x^2+1} \leq \frac{1}{2}}$$

greenified

Proof. Let $x \in \mathbb{R}$ be arbitrary. Then $\boxed{(|x|-1)^2 \geq 0}$. Hence $\boxed{|x|^2 - 2|x| + 1 \geq 0}$. Consequently $\boxed{x^2+1 \geq 2|x|}$. Therefore $\boxed{\frac{|x|}{x^2+1} \leq \frac{1}{2}}$.

Based on the intermezzo we have

$$\boxed{\frac{|x|}{x^2+1} \leq \frac{1}{2}} \quad \text{and} \quad \boxed{\frac{|a|}{a^2+1} \leq \frac{1}{2}}.$$ G3

The transitivity of inequality, G2 and G3 prove DC

daring claim

Not to leave anything in doubt, I will rewrite G2 and G3

$$\frac{|x+a|}{(x^2+1)(a^2+1)} \leq \frac{|x|}{x^2+1} + \frac{|a|}{a^2+1}$$

$$\frac{|x|}{x^2+1} \leq \frac{1}{2}, \quad \frac{|a|}{a^2+1} \leq \frac{1}{2}$$

$$\Rightarrow \frac{|x+a|}{(x^2+1)(a^2+1)} \leq 1$$

Yes, we have just greenified DC

together

Finally, recall G1

$$\left|\frac{1}{x^2+1} - \frac{1}{a^2+1}\right| \leq |x-a| \frac{|x+a|}{(x^2+1)(a^2+1)}$$

$$\left|\frac{1}{x^2+1} - \frac{1}{a^2+1}\right| \leq |x-a|$$

BIN

holds for $\forall a \in \mathbb{R}$ $\forall x \in \mathbb{R}$

This is the best possible BIN!

Now we can state $\delta(\varepsilon) = \varepsilon > 0$

red is green looks good

Now we have to prove:

$$\forall x \in \mathbb{R} \quad \boxed{|x-a| < \varepsilon} \Rightarrow \left| \frac{1}{x^2+1} - \frac{1}{a^2+1} \right| < \varepsilon$$

Recall BIN

$$\boxed{\left| \frac{1}{x^2+1} - \frac{1}{a^2+1} \right| < |x-a|}$$

together

$$\Rightarrow \left| \frac{1}{x^2+1} - \frac{1}{a^2+1} \right| < \varepsilon$$

Proved!