Sequences

Def. A <u>sequence</u> is a function whose domain is the set of positive integers (and codomain the set of real numbers). $s : \mathbb{N} \to \mathbb{R}$.

<u>Examples</u> ⊛ $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, \ldots$ this sequence is given as a list of its values. It is implied that $r_1 = 1$, $r_2 = 2$, $r_3 = 2$, $r_4 = 3$, $r_5 = 3$, $r_6 = 3$, $r_7 = 4$, ...
It is an interesting challange to find a formula for r_n.
In fact $r_n = \left\lfloor \frac{1}{2} + \sqrt{2n} \right\rfloor$, $n \in \mathbb{N}$.

⊛ $1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots$ this sequence is called the powers of 2. $p_n = 2^{n-1}$, $n \in \mathbb{N}$.
In general, for $a \in \mathbb{R}$ $p_n = a^{n-1}$, $n \in \mathbb{N}$ (powers of a).

* Often sequences are given by a <u>recursive</u> formula: The next member of a sequence is given as a formula involving previous members.

<u>Example</u> This is a recursive formula for the powers of a, $a \in \mathbb{R}$.

$$p_1 = 1, \quad p_{n+1} = a p_n, \quad n = 1, 2, 3, \ldots$$

<u>Example</u> $x_1 = 2, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \quad n = 1, 2, 3, \ldots$

$$\underbrace{x_2}_{n=1} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}, \quad \underset{n=2}{x_3} = \frac{3/2}{2} + \frac{1}{3/2} = \frac{3}{4} + \frac{2}{3} = \frac{17}{12}, \ldots$$

$x_4 =$ (computers love recursive formulas)

$$x_1 = 2, \; \underset{1.5}{x_2 = \frac{3}{2}}, \; \underset{\approx 1.41667}{x_3 = \frac{17}{12}}, \; \underset{\approx 1.41421}{x_4 = \frac{577}{408}}, \ldots$$

An interesting question is: Do the numbers $x_1, x_2, x_3, \ldots$ approach some limit L? Does the limit $x_n \to L$ exist as $n \to +\infty$?

As you can guess from the approximate values

$$x_n \to \sqrt{2} \text{ as } n \to +\infty.$$

We will prove this later.

Two famous sequences:

$$x_n = \left(1+\frac{1}{n}\right)^n, \quad n \in \mathbb{N}$$ (sequence given by a formula)

$$t_1 = 2, \quad t_{n+1} = t_n + \frac{1}{(n+1)!}, \quad n = 1, 2, 3, \ldots$$

Calculate: $t_2 = 2 + \frac{1}{2!}$, $t_3 = 2 + \frac{1}{2!} + \frac{1}{3!}$, $t_4 = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$

$$t_5 = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$

$$t_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$ (the sum of reciprocal of the factorials!)

An amazing fact

$$\left(1+\frac{1}{n}\right)^n \to e \quad (n \to +\infty)$$

Proved earlier

$$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!} \to e \quad (n \to +\infty)$$

Convergence of a sequence

Definition very similar to: $\lim_{x \to +\infty} f(x) = L$

A sequence $s : \mathbb{N} \to \mathbb{R}$ converges to $L \in \mathbb{R}$ as $n \to +\infty$ if the following condition is satisfied:

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N} \quad n > N(\varepsilon) \Rightarrow |s_n - L| < \varepsilon$$

If s. converges to L we write $\lim_{n\to+\infty} s_n = L$

<u>Example</u> (very important) Let $r \in (-1,1)$

Then $\lim_{n\to+\infty} r^n = 0$.

<u>Proof.</u> Let $\varepsilon > 0$ be arbitrary. Need to find $N(\varepsilon)$ such that $\forall n \in \mathbb{N}$ $n > N(\varepsilon) \Rightarrow \underbrace{|r^n - 0| < \varepsilon}_{\text{as before solve for } n}$.

Simplify: $|r^n - 0| = |r^n| = |r|^n$.

Solve $|r|^n < \varepsilon$. Take ln of both sides.

$n \ln|r| < \ln \varepsilon$ here $\ln|r| < 0$

so $n > \ln\varepsilon / \ln|r|$

In fact, since $\ln|r| < 0$ we have the following equivalences

$$|r|^n < \varepsilon \iff n \ln|r| < \ln\varepsilon \iff n > \frac{\ln\varepsilon}{\ln|r|}$$

Hence, we can set $N(\varepsilon) = \dfrac{\ln\varepsilon}{\ln|r|}$.

Then $n \in \mathbb{N}$ and $n > \dfrac{\ln\varepsilon}{\ln|r|} \Rightarrow n\ln|r| < \ln\varepsilon \Rightarrow \ln|r|^n < \ln\varepsilon$

$$\Rightarrow |r|^n < \varepsilon$$

$$\Rightarrow |r^n - 0| < \varepsilon$$