# Sequences (Infinite series defined at the website)

Please also see the post at the class website on this day (google curgus wwu)

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# Sequences

A sequence is a function whose domain is $\mathbb{N}$.

$s : \mathbb{N} \to \mathbb{R}$, $s_1, s_2, s_3, \ldots$

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ $\quad s_1 = s(1)$, $s_2 = s(2)$

Sometimes we start from 0, $\mathbb{N}_0 = \{0, 1, \ldots\}$

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* Two kinds of sequences:

1st kind: A sequence given by a formula

$a_n = n^2, \forall n \in \mathbb{N}$

$b_n = \frac{1}{n}, \forall n \in \mathbb{N}$

$n \in \mathbb{N}$ $r_n = \left\lfloor \frac{1}{2} + \sqrt{2n} \right\rfloor$, $r_1 = 1, r_2 = 2, r_3 = 2, r_4 = 3, r_5 = 3, r_6 = 3,$

$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, \ldots$

2nd kind of sequences: Sequences given by a recursive formula:

* $x_1 = 2, \quad x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \quad n = 1, 2, 3, \ldots$

* $y_1 = 1, \quad \underset{\text{next}}{y_{n+1}} = \underbrace{y_1 + \cdots + y_n}_{\text{previous terms}}, \quad n = 1, 2, 3, \ldots$

$y_1 = 1, y_2 = 1, y_3 = 2, y_4 = 4, y_5 = 8, y_6 = 16, y_7 = 32, \ldots$

* $f_1 = 1, \quad f_{n+1} = (n+1) * f_n, \quad n = 1, 2, 3, \ldots$

$f_1 = 1, f_2 = 2 \cdot 1, f_3 = 3 \cdot 2 \cdot 1, \ldots$

$n!$ definition of n-factorial

$f_0 = 1$

$0! = 1$

$\circledast$ $T_0 = 1 = \frac{1}{0!}$, $T_n = T_{n-1} + \frac{1}{n!}$, $n = 1, 2, \ldots$

$T_1 = \frac{1}{0!} + \frac{1}{1!}$, $T_2 = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!}$, $T_3 = \sum_{k=0}^{3} \frac{1}{k!}$, $\ldots$, $T_n = \underbrace{\sum_{k=0}^{n} \frac{1}{k!}}_{\text{partial sum}}$

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$s : \mathbb{N} \to \mathbb{R}$ popular way of writing sequences is

$\{s_n\}_{n=1}^{\infty} = \{s_n\}_{n \in \mathbb{N}}$

We are interested in limits of sequences:

$\boxed{\lim_{n\to+\infty} s_n = L}$ means:

Def: $\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{R}$ s.t.

$\forall n \in \mathbb{N} \quad n > N(\varepsilon) \Rightarrow |s_n - L| < \varepsilon$

Recall def. of $\boxed{\lim_{x\to+\infty} f(x) = L}$ for $f : [1, +\infty) \to \mathbb{R}$ means

$\forall \varepsilon > 0$ $\exists X(\varepsilon) \geq 1$ s.t.

$x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$

just takes care of the domain

**Theorem** Let $f : [1,+\infty) \to \mathbb{R}$ be a function

Assume that $\lim_{x\to+\infty} f(x) = L$ and

assume that $a_n = f(n)$ $\forall n \in \mathbb{N}$.

TRIVIAL

not to overuse stat.

**Claim:** $\boxed{\lim_{n\to+\infty} a_n = L}$

**Theorem** Let $f : (0,1] \to \mathbb{R}$ be function

Assume: ① $\lim_{x\downarrow 0} f(x) = L$ ② $a_n = f(1/n)$ $\forall n \in \mathbb{N}$

**Claim:** $\boxed{\lim_{n\to+\infty} a_n = L}$

Example (super important) $n \mapsto r^n, \forall n \in \mathbb{N}$

$r \in (-1, 1)$

$$\lim_{n \to +\infty} r^n = 0.$$

Prove it! (punctuation, not factorial)

A very important number is 0!

~~ambiguous~~ sentence

? wrong ?

# Abstract theorems about LIMITS

ALGEBRA of LIMITS

LIMITS respect ORDER (among reals)