Convergent sequences RESPECT the algebra of real numbers!

They also respect the order among real numbers!

May 14, 2020

Convergent Sequences

Def $s : \mathbb{N} \to \mathbb{R}$. s converges

$\exists L \in \mathbb{R}$ s.t.

$\forall \varepsilon > 0 \; \exists N(\varepsilon) \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ we have

$$n > N(\varepsilon) \Rightarrow |s_n - L| < \varepsilon$$

$$\lim_{n \to +\infty} s_n = L \quad \text{or} \quad s_n \to L \; (n \to +\infty)$$

Theory about convergent seq.

Algebra of convergent sequences

↳ (algebra of real numbers)

addition, mult., subtraction, division

Thm: Let $a, b : \mathbb{N} \to \mathbb{R}$.

Assume $\boxed{\lim_{n\to+\infty} a_n = K}$, $\boxed{\lim_{n\to+\infty} b_n = L}$

Then we have:

(A) If $\boxed{c_n = a_n + b_n}$, $\forall n \in \mathbb{N}$, then $c : \mathbb{N} \to \mathbb{R}$ $\boxed{\text{converges}}$ and $\boxed{\lim_{n\to+\infty} c_n = K + L}$

(B) If $c_n = a_n \cdot b_n \ \forall n \in \mathbb{N}$, then $c: \mathbb{N} \to \mathbb{R}$ converges and $\lim\limits_{n \to +\infty} c_n = K \cdot L$.

(C) If $b_n \neq 0 \ \forall n \in \mathbb{N}$ and $c_n = \dfrac{a_n}{b_n}$ and $L \neq 0$, then $c: \mathbb{N} \to \mathbb{R}$ converges and $\lim\limits_{n \to +\infty} c_n = \dfrac{K}{L}$

Convergent sequences RESPECT algebra of real numbers!

Thm Let $a : \mathbb{N} \to \mathbb{R}$, $b : \mathbb{N} \to \mathbb{R}$.

Assume $\exists n_0 \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$ $n \geq n_0 \Rightarrow a_n \leq b_n$. G1

$\lim\limits_{n \to +\infty} a_n = K$ G2 and $\lim\limits_{n \to +\infty} b_n = L$. G3

Then $\boxed{K \leq L}$

Convergent sequences respect ORDER among real numbers

Proof. G2 $\forall \varepsilon > 0$ $\exists N_a(\varepsilon) \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $n > N_a(\varepsilon) \Rightarrow |a_n - K| < \varepsilon$

G3 $\forall \varepsilon > 0\ \exists N_b(\varepsilon) \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$

$$n > N_b(\varepsilon) \Rightarrow |b_n - L| < \varepsilon$$

My hope for ORDER is G1, order between $a_n \le b_n$

In G2 and G3 a_n and b_n are concealed under the absolute value sign. How do we uncover a_n and b_n? We use the following equivalencies: $|a_n - K| < \varepsilon \iff -\varepsilon + K < a_n < K + \varepsilon$

$|b_n - L| < \varepsilon \iff -\varepsilon + L < b_n < L + \varepsilon$

explanation for the equivalences using I-5 through Bellingham

BBB - Blaine Bellingham - Burlington.

you on I-5

$|a_n - K| < \varepsilon$ ← tiny

Blaine. Bell. 25 miles

$|y - a| < c$

Bell. 25

Between Burlington & Blaine

$$K-\varepsilon < a_n < K+\varepsilon$$

$$L-\varepsilon < b_n < L+\varepsilon$$

$$a-c < y < a+c$$

$\Rightarrow$ Purl. Ball. Blaise

$\Leftrightarrow |a_n - K| < \varepsilon$

$\Leftrightarrow |b_n - L| < \varepsilon$

Let us recall all green stuff that we have:

G1 $\forall n \in \mathbb{N}$ $\boxed{n \geq n_0} \Rightarrow \boxed{a_n \leq b_n}$ B1

G2 $\forall \varepsilon > 0$ $\exists N_a(\varepsilon) \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $\boxed{n > N_a(\varepsilon)} \Rightarrow \boxed{-\varepsilon + K < a_n < K + \varepsilon}$ B2

G3 $\forall \varepsilon > 0$ $\exists N_b(\varepsilon) \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $n > N_b(\varepsilon) \Rightarrow \boxed{-\varepsilon + L < b_n < L + \varepsilon}$ B3

We have three boxes with inequalities: B1, B2, B3

We need all three boxes to be true.

Let $\varepsilon > 0$ be arbitrary. Let $n \in \mathbb{N}$ be such that

$$n > \max\{n_0, N_a(\varepsilon), N_b(\varepsilon)\}$$

Since $n > n_0$ it is true that $\boxed{a_n \le b_n}$

Since $n > N_a(\varepsilon)$ it is true that $\boxed{-\varepsilon + K < a_n}$

Since $n > N_b(\varepsilon)$ it is true that $\boxed{b_n < L + \varepsilon}$.

By the transitivity property of order we conclude

$$\boxed{-\varepsilon + K < a_n \le b_n < L + \varepsilon} \Rightarrow \boxed{K - L < 2\varepsilon}$$

Thus we have proved that

$$\boxed{\forall \varepsilon > 0 \quad K - L < 2\varepsilon.}$$

The last green boxed statement implies $\boxed{K - L \le 0}$ Hence $\boxed{K \le L}$

$$\boxed{\forall \varepsilon > 0 \quad \alpha \le \varepsilon \Rightarrow \alpha \le 0}$$

To prove consider the contrapositive

Contrapositive : $\boxed{\alpha > 0 \Rightarrow \exists \varepsilon > 0 \text{ s.t. } \alpha > \varepsilon.}$? Trivial statement.

set $\varepsilon = \alpha/2$