A detailed proof of the Monotone Convergence Theorem

My thoughts on writing:
Think through writing.
Learn through writing.
Write, for the audience of one:
Yourself.

May 19, 2020

Learn through writing!

Think through writing!

Write for yourself!

my thoughts
the scrapbook
to write in

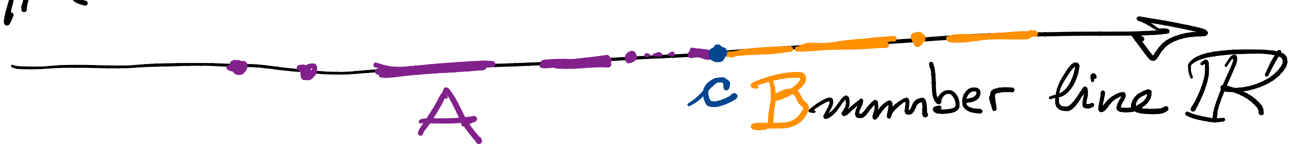
The formulation of the Completeness Axiom below is from a book by

Zorich

The Completeness Axioms of $\mathbb{R}$

If $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, $A \neq \emptyset$, $B \neq \emptyset$ and $\forall a \in A\ \forall b \in B$ we have $a \leq b$, then

$$\exists c \in \mathbb{R} \text{ s.t. } a \leq c \leq b \ \forall a \in A \ \forall b \in B$$



The Monotone Convergence Theorem

If a sequence is monotonic and bounded, then it converges.

Proof. Case 1. A sequence is non-decreasing and bounded above.

Let $s: \mathbb{N} \to \mathbb{R}$ be non-decreasing and bdd above.

That is $\forall n \in \mathbb{N}$ $\boxed{s_n \leq s_{n+1}}$ $(s_1 \leq s_2 \leq \dots \leq s_n \leq s_{n+1} \leq \dots$

and $\exists M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$ $\boxed{s_n \leq M}$

What is RED?

$\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$ $\exists N(\varepsilon) \in \mathbb{R}$ s.t.

$\forall n \in \mathbb{N}$ we have $n > N(\varepsilon) \Rightarrow |s_n - L| < \varepsilon$

This smells like CA! (Completeness Axiom)

So, set

$$A = \{ s_n : n \in \mathbb{N} \} \quad (\text{range of } s : \mathbb{N} \to \mathbb{R}),$$

$$B = \{ b \in \mathbb{R} : b \text{ is an upper bound for } s \}$$

Clearly $A \neq \emptyset$, $B \neq \emptyset$ since $M \in B$.

Clearly $\forall a \in A \ \forall b \in B$ we have $a \leq b$

I $a \in A$, then $\underset{\text{for some } n \in \mathbb{N}}{a = s_n} \leq b \in B$ since b is an upper bound for s.

Thus All the hypothesis of CA are satisfied

Therefore $\exists\, c \in \mathbb{R}$ such that

$$\boxed{\underline{\forall n \in \mathbb{N}\ \forall b \in B} \quad \boxed{\underbrace{s_n}_{\text{any element of } A} \leq c \leq \underbrace{b}_{\text{any upper bound for } A}}} \quad G1$$

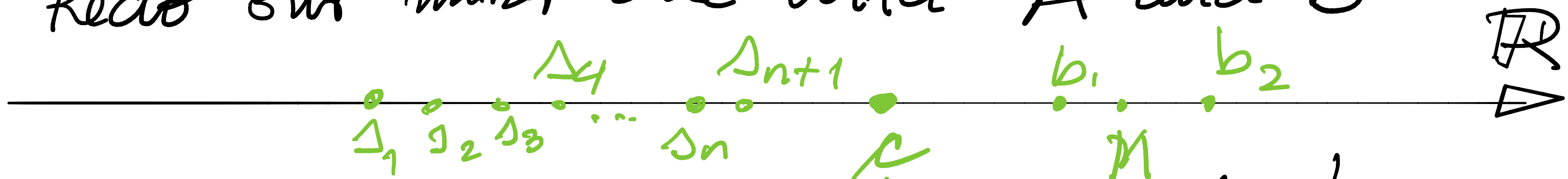
By G1 c is an upper bound for the sequence s.

Thus $c \in B$ (all upper bounds for s)

(c is called the least upper bound for s)

(In fact we have that $c = \min B$
c is the minimum of B)

Redo our number line with A and B



c is special if it is the least upper bound for A

$c - \varepsilon$, $\varepsilon > 0$

this is NOT an upper bound.

Now we are ready to address RED!

Set $L = c$. smells like a solution.

Let $\varepsilon > 0$ be arbitrary. Then $c - \varepsilon < c$ but $c \leq b$ for all b upper bounds for A.

therefore $c - \varepsilon$ is NOT an upper bound for A.

$c - \varepsilon$ is NOT an upper bound for Λ.

How do you say this in Mathish language?

This is how:

$\exists N(\varepsilon) \in \mathbb{N}$ such that

$$\boxed{c - \varepsilon < \Lambda_{N(\varepsilon)}} \qquad G2$$

Now if I take $n \geq N(\varepsilon)$, since Λ_n is non-decreasing, I have

$$G3 \quad \boxed{\Lambda_{N(\varepsilon)} \leq \Lambda_n} \quad \forall n \geq N(\varepsilon)$$

From G2 & G3 we conclude that

$$\forall n \in \mathbb{N} \quad n \geq N(\varepsilon) \Rightarrow c - \varepsilon < s_n.$$

That is: $\boxed{\forall n \in \mathbb{N} \quad n \geq N(\varepsilon) \Rightarrow c - s_n < \varepsilon}$ G4

But we know from G1 that $\forall n \in \mathbb{N} \quad c \geq s_n$.

Therefore $\forall n \in \mathbb{N} \quad |s_n - c| = c - s_n$.

Thus G4 can be rewritten as

$$\forall n \in \mathbb{N} \quad n \geq N(\varepsilon) \Rightarrow |s_n - c| < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we have proved that

$$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N} \text{ such that}$$

$$\forall n \in \mathbb{N} \quad n \geq N(\varepsilon) \Rightarrow |s_n - c| < \varepsilon.$$

This proves that $L = c$ is the limit of $s : \mathbb{N} \to \mathbb{R}$. We proved

$$\lim_{n \to +\infty} s_n = c.$$