Harmonic Series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n}$

sequence of

Study of the Harmonic Numbers

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N}$$

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* increasing
* not bdd above

$s: \mathbb{N} \to \mathbb{R}$ $\boxed{\lim_{n\to\infty} s_n = L}$

$p \in \mathbb{N}$ then $\boxed{\lim_{n\to\infty} s_{n+p} = L}$.

Formally s_{n+p}, $n \in \mathbb{N}$ is "another" sequence

call it $b: \mathbb{N} \to \mathbb{R}$, $b_n = s_{n+p}$ $\forall n \in \mathbb{N}$.

$$\forall \varepsilon > 0 \quad \exists N(\varepsilon) \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N} \quad n > N(\varepsilon) \overset{\text{GI}}{\Rightarrow} |s_n - L| < \varepsilon$$

$$\forall \varepsilon > 0 \quad \exists M(\varepsilon) \in \mathbb{R} \text{ s.t. } \forall n \in \mathbb{N} \quad n > M(\varepsilon) \Rightarrow |s_{n+p} - L| < \varepsilon$$

Let $\varepsilon > 0$ be arbitrary. Set $M(\varepsilon) = N(\varepsilon) - p$

smells like a solution!

Now prove

Let $n \in \mathbb{N}$ be such that $\boxed{n > M(\varepsilon) = N(\varepsilon) - p}$.

then $\boxed{n+p > N(\varepsilon)}$. By GI see above

I deduce $\boxed{|s_{n+p} - L| < \varepsilon}$

The big red box above has completely been greenified!

Let us go back to <u>Infinite Series</u>

Another Specific famous Infinite Series is the <u>Harmonic Series</u>:

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Its partial Sums are called <u>Harmonic Numbers</u>

$n \in \mathbb{N}$ $$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

$$H_1 = 1$$

$$H_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{2} + \frac{1}{3} = \frac{9+2}{6} = \frac{11}{6}$$

$\vdots$

$\forall n \in \mathbb{N}$ $\boxed{H_{n+1} - H_n = \frac{1}{n+1} > 0}$

Hence $\forall n \in \mathbb{N}$ $\boxed{H_n < H_{n+1}}$.

Thus, the sequence of harmonic numbers is INCREASING.

By the MCT, the sequence of harmonic numbers converges if and only if it is bdd above.

The sequence of Harmonic numbers is NOT bdd above.

I will greenify this statement next.

This greenification is a beautiful reasoning.

We consider the Harmonic numbers with indexes that are the powers of 2: 1, 2, 4, 8, 16, 32, ...

$$H_{2^0} = H_1 = 1$$

this is reasoning.

Pizza-Party

We want less pizza in this context

$$H_{2^1} = H_2 = 1 + \frac{1}{2}$$

$$H_{2^2} = H_4 = H_2 + \frac{1}{3} + \frac{1}{4} \geq H_2 + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2 \cdot \frac{1}{2}$$

$$H_{2^3} = H_8 = H_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq H_4 + 4 \cdot \frac{1}{8} \geq 1 + 2 \cdot \frac{1}{2} + \frac{1}{2} = 1 + 3 \cdot \frac{1}{2}$$

$$H_{2^4} = H_{16} = H_8 + \frac{1}{9} + \dots + \frac{1}{16} \geq H_8 + 8 \cdot \frac{1}{16} \geq 1 + 3\frac{1}{2} + \frac{1}{2} = 1 + 4 \cdot \frac{1}{2}$$

this is a recursive proof of Pizza-Party recursion

$$H_{2^m} = H_{2^{m-1}} + \underbrace{\frac{1}{2^{m-1}+1} + \dots + \frac{1}{2^m}}_{2^{m-1}} \geq H_{2^{m-1}} + \frac{1}{2} \geq 1 + (m-1)\frac{1}{2} + \frac{1}{2} = 1 + m\frac{1}{2}$$

Thus we recursively proved (Mathematical Induction)

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$$\boxed{\forall m \in \mathbb{N}_0 \quad H_{2^m} \geq 1 + \frac{m}{2}}$$

B/X

not bdd above

$\exists M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $a_n \leq M$

I want to prove that

$$\boxed{\forall M \in \mathbb{R} \quad \exists n_M \in \mathbb{N} \text{ s.t. } H_{n_M} > M}$$

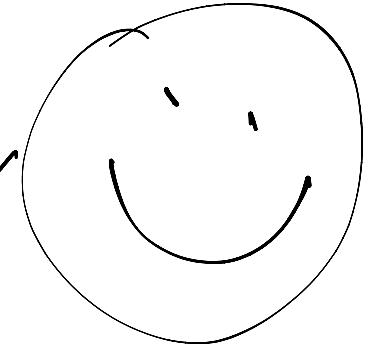
hard

Greenify this RED BOX

Let $M \in \mathbb{R}$ be arbitrary.

$H_{n_M} > M$ hard.

Hard to solve for n_M

$H_{2^m} > M$ 

$$1 + \frac{m}{2} > M$$

$\Updownarrow$

easy to solve for m

Solve it: $\frac{m}{2} > M-1 \iff m > 2M-2$

one solution is $m = \lfloor 2M-1 \rfloor$

Since we need $m \in \mathbb{N}_0$, we must have $2M-1 \geq 0$, that is $M \geq 1/2$.

But, for $M < 1/2$ we can take $m = 0$. Thus

$\forall M \in \mathbb{R}$ setting $n_M = 2^{\max\{\lfloor 2M-1 \rfloor, 0\}}$

will lead to $\boxed{H_{n_M} > M}$. needs to be greenified.

Set $m_M = \max\{\lfloor 2M-1 \rfloor, 0\}$. Then

$$H_{2^{m_M}} \underset{\text{proved above}}{\geq} 1 + \frac{m_M}{2} \underset{\text{definition of } m_M}{\geq} 1 + \frac{\lfloor 2M-1 \rfloor}{2} \underset{\text{property of the floor function}}{>} 1 + \frac{2M-2}{2} \underset{\text{algebra}}{=} M$$