A squeeze for two common sequences that converge to e

The following two sequences are commonly used to define the number e:

$$S_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad P_n = \left(1 + \frac{1}{n}\right)^n, \qquad n \in \mathbb{N}.$$

Two things: converge and have the same limit.

Here $\mathbb{N}$ denotes the set of positive integers.

In this note we give a direct proof that $\{S_n\}$ and $\{P_n\}$ converge to the same limit. The main tool in our proof is the Squeeze Theorem, which is probably the easiest to prove among the limit theorems. However, to use the Squeeze Theorem, we need to establish a relevant squeeze, which is the main result of this note.

Surprisingly, many elementary mathematical analysis textbooks do not include a proof that $\{S_n\}$ and $\{P_n\}$ converge to the same limit. The proofs in the classical book [4, Theorem 3.31], and in more recent [1, Proposition 3.3.1] and [3, Appendix 2] all use a limit theorem which they do not prove.

The sequence $\{S_n\}$ converges *greenify* For completeness we give the standard proof that $\{S_n\}$ is bounded above by 3.

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Clearly, $S_1 = 2 < 3$ and as $1/k! \leq 1/((k-1)k)$ for all $k > 1$ we have that

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} \leq 2 + \sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k}\right) = 3 - \frac{1}{n},$$

for all $n > 1$. Thus $S_n < 3$ for all $n \in \mathbb{N}$. Since $\{S_n\}$ is increasing, it converges by the Monotone Convergence Theorem.

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The squeeze The squeeze that we announced earlier is:

$$S_n - \frac{3}{2n} \leq P_n \leq S_n \qquad \text{for all} \qquad n \in \mathbb{N}. \tag{1}$$

Applying the Squeeze Theorem to (1) shows that $\{P_n\}$ converges to the same limit as $\{S_n\}$, namely e. *The last sentence is brief. To use squeeze, you must prove $\lim_{n\to\infty}(S_n - 3/2n) = \lim_{n\to\infty} S_n$. Only then squeeze T can be applied.*

Proof of the squeeze Since (1) is true for $n \leq 2$, consider $n > 2$. Our proof of (1) is a succession of four steps, each suggesting the next one.

1. The **binomial theorem** yields an expanded expression for P_n: *please review*

$$P_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{1}{n^k} = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!}\frac{n!}{(n-k)!\,n^k}. \tag{2}$$

2. For $k \in \{2, \ldots, n\}$, we **rewrite the coefficient** with $1/k!$ in (2) as the product of $k - 1$ factors:

$$\frac{n!}{(n-k)!\,n^k} = \frac{(n-1)\cdots(n-k+1)}{n^{k-1}} = \left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right). \tag{3}$$

3. An **upper bound** for the product in (3) is clearly 1, so we look for its **lower bound** next. We proceed recursively. At each step, in some sense, we turn a product into a smaller sum.

For $k = 2$ the product in (3) has only one term and obviously $(1 - 1/n) \geq 1 - 1/n$. For $k = 3$, we expand the product and drop a positive term to get:

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) = 1 - \frac{1+2}{n} + \frac{1 \cdot 2}{n} > 1 - \frac{1+2}{n}.$$

For $k = 4$, we multiply both sides above by $\left(1 - \frac{3}{n}\right)$, expend the product on the right and drop a positive term to get: *(skipped step)*

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) > \left(1 - \frac{1+2}{n}\right)\left(1 - \frac{3}{n}\right) > 1 - \frac{1+2+3}{n}.$$

Repeating this process a total of $k - 1$ times and, at the end, using the familiar formula $1 + 2 + \cdots + (k - 1) = (k - 1)k/2$ (whose history is given in the impressive collection [2]) yields

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) > 1 - \frac{1 + \cdots + (k-1)}{n} = 1 - \frac{(k-1)k}{2n}.$$

In conclusion, for the product in (3) we have

This is the key of the proof.

$$1 - \frac{(k-1)k}{2n} < \frac{n!}{n^k(n-k)!} < 1 \quad \text{for all} \quad k \in \{2, \ldots, n\}. \tag{4}$$

4. The inequalities in (4) are applied to the most right expression in (2) to **establish the inequalities** for P_n:

$$1 + 1 + \sum_{k=2}^{n} \frac{1}{k!}\left(1 - \frac{(k-1)k}{2n}\right) < P_n < 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot 1 = S_n. \tag{5}$$

A simplification of the left-hand side of (5) leads to

$$\sum_{k=0}^{n} \frac{1}{k!} - \sum_{k=2}^{n} \frac{1}{k!}\frac{(k-1)k}{2n} = S_n - \frac{1}{2n}\sum_{k=2}^{n} \frac{1}{(k-2)!} = S_n - \frac{1}{2n}S_{n-2}.$$

subtract more, then you have less

Further, since $S_{n-2} < 3$, we have

$$S_n - \frac{1}{2n}S_{n-2} > S_n - \frac{3}{2n}.$$

Consequently, the left-hand side of (5) is greater than $S_n - 3/(2n)$. This proves the squeeze in (1) for all $n > 2$ and completes the proof.

The squeeze in (1) was red. Here it is greenified. ☺

References

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4. Walter Rudin, *Principles of Mathematical Analysis*. Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill, 1976.

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