Proofs of Prop. 5.1

and more ....

Prop. 5.1. $a, b, c \in \mathbb{R}$

$$\boxed{\forall x \in \mathbb{R} \quad ax^2+bx+c \geq 0} \Rightarrow \boxed{a \geq 0}$$

hypothesis

conclusion

CONTRAPOSITIVE

($p \Rightarrow q$ contrapo. is $\neg q \Rightarrow \neg p$)

$$\boxed{a < 0} \Rightarrow \boxed{\exists x \in \mathbb{R} \quad ax^2+bx+c < 0}$$

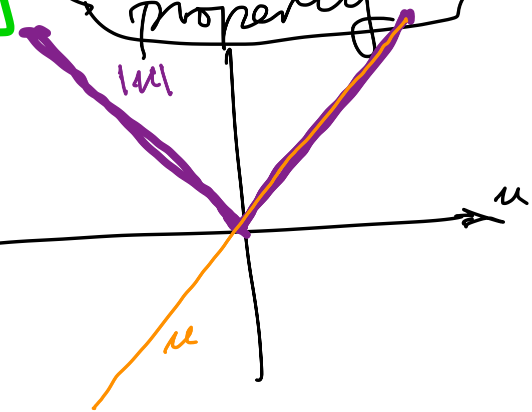
$a, b, c \in \mathbb{R}$

Background Knowledge – Prerequisite

BK

(1) $\forall u \in \mathbb{R} \quad u \le |u|$

(we will prove this soon properly)



(2) $\forall x, y, z \in \mathbb{R}$

$(x \le y) \wedge (z \ge 0) \Rightarrow xz \le yz$

(3) $\forall a, b, c, d \in \mathbb{R}$

$(a \le b) \wedge (c \le d) \Rightarrow a+c \le b+d$

(4) $\forall a, b, c \in \mathbb{R} \quad (a \le b) \wedge (b \le c) \Rightarrow a \le c$

Transitive property of order of reals

Now my proof of P. 5.1. Let $a, b, c \in \mathbb{R}$.

Assume $\boxed{a < 0}$.

The goal is to create $x \in \mathbb{R}$ such that

$$ax^2 + bx + c < 0.$$

I cultivate my interests. { Aleeya helped me with this formulation.

In the notes I explained my reasoning toward getting this formula

In fact, to lower the suspense, I tell you who did it:

$$x_0 = 1 - \frac{|b| + |c|}{a}$$

$$x = x_0$$

Now why is x_0 correct?

1st $\quad ax_0^2 + |b|x_0 + |c|x_0 =$ BK

$\underset{BK}{=} (ax_0 + |b| + |c|)x_0 \quad \left(x_0 = 1 - \frac{|b|+|c|}{a}\right)$

$\underset{BK}{=} (a - \cancel{|b|} - \cancel{|c|} + \cancel{|b|} + \cancel{|c|})\left(1 - \frac{|b|+|c|}{a}\right)$

$\underset{BK}{=} a\left(1 - \frac{|b|+|c|}{a}\right) = \underline{a - |b| - |c| < 0}$

Thus $\boxed{ax_0^2 + |b|x_0 + |c|x_0 < 0}$

2nd I see that $x_0 \geq 1$. Why?

$$|b| \geq 0 \wedge |c| \geq 0 \overset{\text{use BK3}}{\Longrightarrow} |b|+|c| \geq 0$$

$$a < 0 \Rightarrow -a > 0 \Rightarrow -\frac{1}{a} > 0$$

$$\left(-\frac{1}{a} > 0\right) \wedge \left(|b|+|c| \geq 0\right) \overset{\text{BK2}}{\Longrightarrow} -\frac{|b|+|c|}{a} \geq 0$$

$$-\frac{|b|+|c|}{a} \geq 0 \overset{\text{BK3}}{\Longrightarrow} \boxed{1 - \frac{|b|+|c|}{a} \geq 1}$$

3rd

$$\left(b \leq |b|\right) \wedge \left(x_0 \geq 1\right) \underset{\text{BK2}}{\Longrightarrow} \boxed{b x_0 \leq |b| x_0}$$

$$x_0 > 0$$

$$\left(1 \leq x_0\right) \wedge \left(|c| \geq 0\right) \underset{\text{BK2}}{\Longrightarrow} |c| \leq |c| x_0$$

$$\left(c \leq |c|\right) \wedge \left(|c| \leq |c| x_0\right) \underset{\text{BK4}}{\Longrightarrow} \boxed{c \leq |c| x_0}$$

We proved 1st $\boxed{ax_0^2+|b|x_0+|c|x_0<0}$

3rd $(bx_0\leq |b|x_0)\wedge(c\leq |c|x_0)\Rightarrow$

$$\Rightarrow bx_0+c\leq |b|x_0+|c|x_0$$

$$\Rightarrow \boxed{ax_0^2+bx_0+c\leq ax_0^2+|b|x_0+|c|x_0}$$

By the last boxed inequality, the inequality 1st and BK41 we deduce

$$ax_0^2+bx_0+c<0$$

Now I will try to calculate $ax_0^2+bx_0+c$ for $x_0=1-\frac{|b|+|c|}{a}$.

$$a\left(1-\frac{|b|+|c|}{a}\right)^2+b\left(1-\frac{|b|+|c|}{a}\right)+c=\cancel{a}\,\frac{(a-|b|-|c|)^2}{a^{\cancel{2}}}+b\,\frac{a-|b|-|c|}{a}+c$$

$$=\frac{1}{a}\left((a-|b|-|c|)^2+ba-b|b|-b|c|+ac\right)$$

$$=\frac{1}{a}\left(a^2+|b|^2+|c|^2-2a|b|-2a|c|+2|b||c|+ba-b|b|-b|c|+ac\right)$$

$$=\underbrace{\frac{1}{a}}_{<0}\left(\underbrace{a^2}_{>0}+\underbrace{|b|(|b|-b)}_{\geqslant 0}\underbrace{-a(2|b|-b)}_{\geqslant 0}\underbrace{-a(2|c|-c)}_{\geqslant 0}+\underbrace{|c|(2|b|-b)}_{\geqslant 0}+\underbrace{|c|^2}_{\geqslant 0}\right)$$

This calculation shows that

$$ax_0^2+bx_0+c<0$$