Definition of Limit

and its Negation

Definition of Limit at $+\infty$

Let $D \subseteq \mathbb{R}$, $L \in \mathbb{R}$, $f : D \to \mathbb{R}$

We say that f has the limit L as $x \to +\infty$

if (I) $\exists X_0 \in D$ s.t. $[X_0, +\infty) \subseteq D$

$\to$ $\boxed{\text{(II) } \forall \varepsilon > 0 \ \exists X(\varepsilon) \geq X_0 \text{ s.t. } x > X(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon}$

the essence of this def.

silent $\forall x$

$$\lim_{x \to +\infty} f(x) = L$$

How do I prove

$$\boxed{\lim_{x \to +\infty} \sin x = 0}$$

is NOT TRUE

I have to prove that the negation of (II) is TRUE.

The structure of (II) is as follows:

$$\forall u \;\; \exists v \;\; \text{s.t.} \;\; P(u,v)$$

$$\exists u \;\text{s.t.}\; \forall v \;\; \neg P(u,v)$$

what is the negation of $p \Rightarrow q$

it is $p \wedge \neg q$

The negation of (II) is

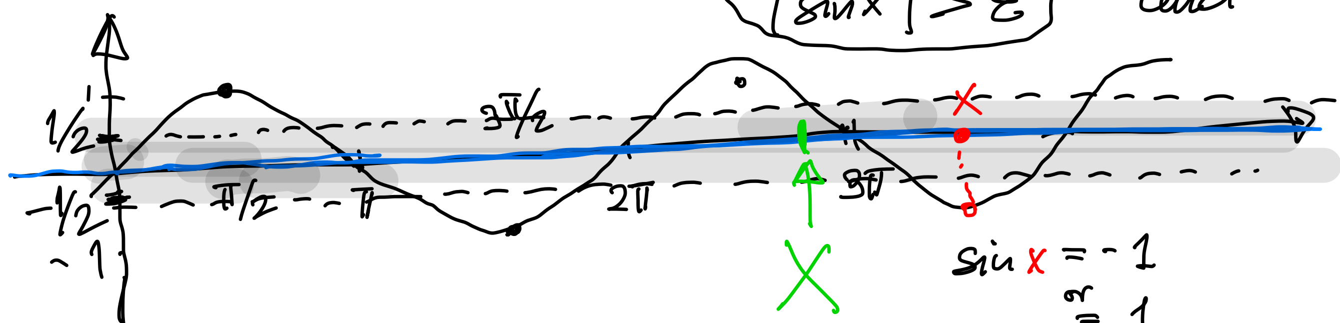
$$\exists \varepsilon > 0 \;\; \forall X \geq X_0 \;\; \exists x > X \;\; \text{s.t.} \;\; \boxed{x > X} \wedge \boxed{|f(x) - L| \geq \varepsilon}$$

$$|\sin x| > \varepsilon$$

and

bad ε

set $\varepsilon = \frac{1}{2}$



Let $X_0 = 0$. Let $\varepsilon = 1/2$. Let $X \geq 0$.

My BK about $\sin x$ is that

$$\boxed{\begin{array}{l} \sin(k\pi) = 0 \quad \forall k \in \mathbb{Z} \\ \sin\left(\frac{\pi}{2} + 2k\pi\right) = 1 \quad \forall k \in \mathbb{Z} \\ \sin\left(\frac{3\pi}{2} + 2k\pi\right) = -1 \quad \forall k \in \mathbb{Z} \end{array}}$$

$\boxed{k \in \mathbb{Z}}$ $\quad x = \frac{\pi}{2} + k\pi > X$

ignore $\pi/2$ solve $k\pi > X$ $\quad k > \frac{X}{\pi}$, $k = \left\lceil \frac{X}{\pi} \right\rceil$

$$\boxed{x = \frac{\pi}{2} + \left\lceil \frac{X}{\pi} \right\rceil \pi. \text{ Then } x > X \text{ and } |\sin x| = 1}$$

Let $\varepsilon = 1/2$. Let $X > 0$ be arb. set $x = \frac{\pi}{2} + \left\lceil \frac{X}{\pi} \right\rceil \pi$. Then $x > X$ and $|\sin x - 0| \geq 1/2$

This proves that $\lim_{x \to +\infty} \sin x = 0$ is NOT true.

Much more is true (than $\lim_{x\to+\infty} \sin x = 0$ is a lie)

$\boxed{\lim_{x\to+\infty} \sin x \text{ DOES NOT EXIST}}$

How to prove this?

We have to prove that $\lim_{x\to+\infty} \sin x = L$ is NOT TRUE for every $L \in \mathbb{R}$.

$\forall L \in \mathbb{R} \ \exists \varepsilon > 0$ s.t. $\forall X > 0 \ \exists x > X$ s.t. $|\sin x - L| \geq \varepsilon$

This is your Goal

Proof. Let $L \in \mathbb{R}$ be arbitrary.

Case 1 $L \geq 0$. Let $\varepsilon = 1/2$.

Let $X > 0$ be arbitrary. I will choose $x > X$ such that $\sin x = -1$. That is

$x = \underbrace{\frac{3\pi}{2}}_{\text{ignore}} + 2k\pi > X$ choose k such that

$k \in \mathbb{Z}$

$2k\pi \geq X$

$k \geq \frac{X}{2\pi}$

$k = \left\lceil \frac{X}{2\pi} \right\rceil$

$\mathbb{Z}$

The following statement is true. For arbitrary $L \geq 0$ and $\varepsilon = \frac{1}{2}$ we have $\forall X > 0$ with $x = \frac{3\pi}{2} + 2\pi \left\lceil \frac{X}{2\pi} \right\rceil$ we have $x > X$ and $|\sin(x) - L| = L + 1 \geq \frac{1}{2}$

This proves that $\lim\limits_{x \to +\infty} \sin x = L$ is not true.

<u>Case 2</u> Assume $L \leq 0$. Set $\varepsilon = 1/2$. Let $X > 0$ be arbitrary set $x = \frac{\pi}{2} + 2\pi \left\lceil \frac{X}{2\pi} \right\rceil$. Then $x > X$ and $|\sin x - L| = 1 - L \geq 1/2$.

This proves that $\lim\limits_{x \to +\infty} \sin x = L$ is not true if $L \leq 0$.

With these two cases, the proof is complete.