Algebra of Limits
(no proofs ☹)

Continuous Functions!

$D \subseteq \mathbb{R}$ $\quad a, K, L \in \mathbb{R}$ $\quad \boxed{\lim_{x \to a} f(x) = K}$

$f : D \to \mathbb{R}$

$g : D \to \mathbb{R}$ $\quad \boxed{\lim_{x \to a} g(x) = L}$

From pre-calculus you are familiar with algebra of functions

$\underbrace{f+g}_{\text{new}}, \ \overset{\text{new}}{f \cdot g}, \ \frac{1}{g}, \ \frac{f}{g}$ (new functions from old)

$f+g : D \to \mathbb{R}$ $\quad \forall x \in D \ (f+g)(x) = f(x) + g(x)$

Theorem (Algebra of Limits)

Assume ① $\lim_{x \to a} f(x) = K$ ② $\lim_{x \to a} g(x) = L$

Then

A If $h = f + g$, then $\lim_{x \to a} h(x) = K + L$

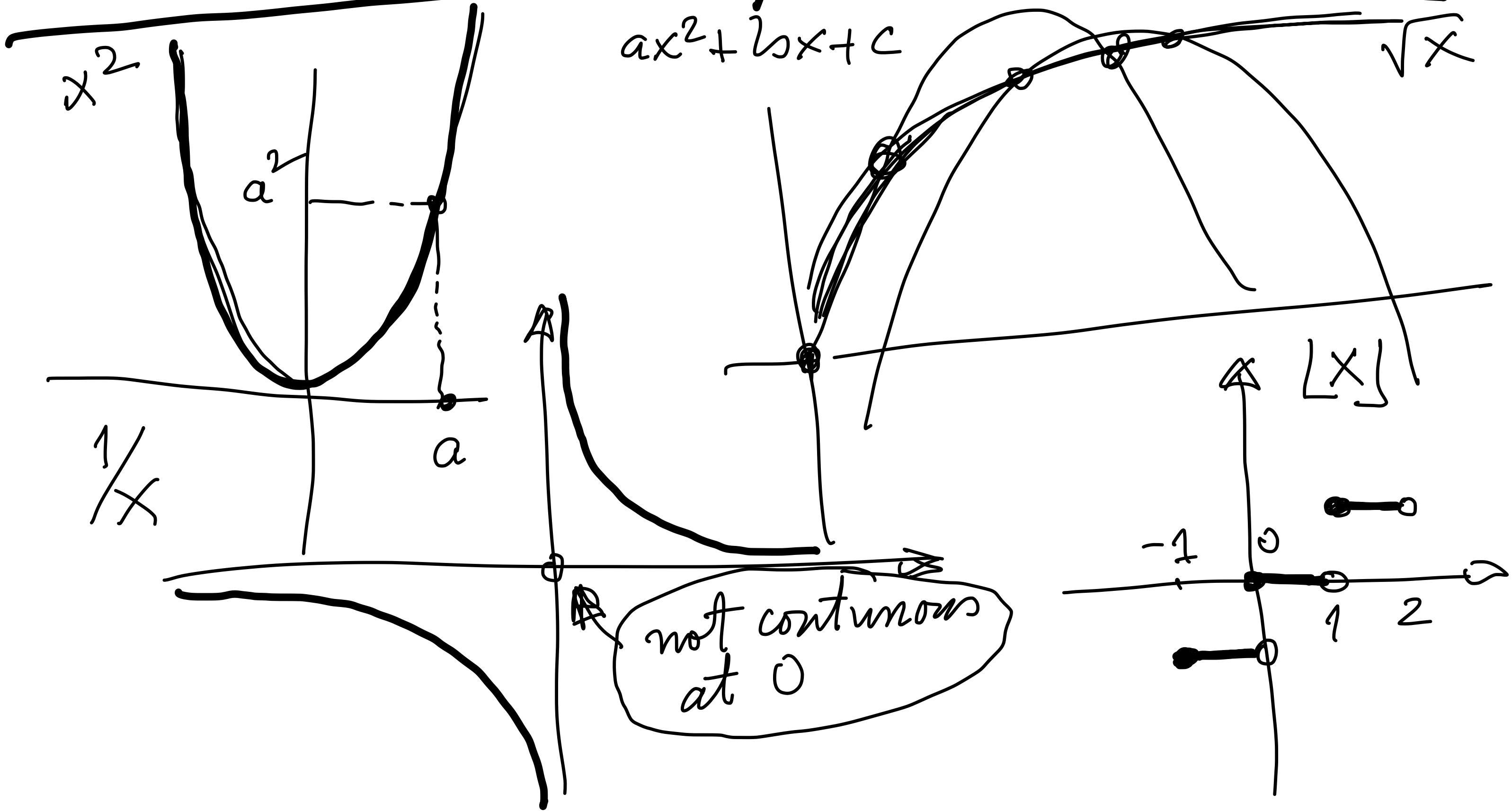
B If $h = f \cdot g$, then $\lim_{x \to a} h(x) = KL$

C If $h = \frac{1}{g}$ and $L \neq 0$, then $\lim_{x \to a} h(x) = \frac{1}{L}$

D If $h = \frac{f}{g}$ and $L \neq 0$, then $\lim_{x \to a} h(x) = \frac{K}{L}$

no time for proofs. All the proofs are in the notes.

Continuous functions



__Definition__ Let $D \subseteq \mathbb{R}$. A function $f: D \to \mathbb{R}$ is continuous at a point c if the following two conditions are satisfied

(I) $c \in D$, that is $f(c)$ is defined

(II) $\boxed{\lim_{x \to c} f(x) = f(c)}$ → ? what does this mean?

This definition hides the true content.

the full def. (from first principles) is

__Def.__ Let $D \subseteq \mathbb{R}$. A function $f: D \to \mathbb{R}$ & $c \in \mathbb{R}$ is continuous at a point c if the following two cond. are satisfied

(I) $\exists \delta_0 > 0$ s.t. $(c-\delta_0, c+\delta_0) \subseteq D$

(II) $\forall \varepsilon > 0 \ \exists \delta(\varepsilon)$ such that $0 < \delta(\varepsilon) < \delta_0$ and

$$|x-c| < \delta(\varepsilon) \Rightarrow |f(x) - f(c)| < \varepsilon$$

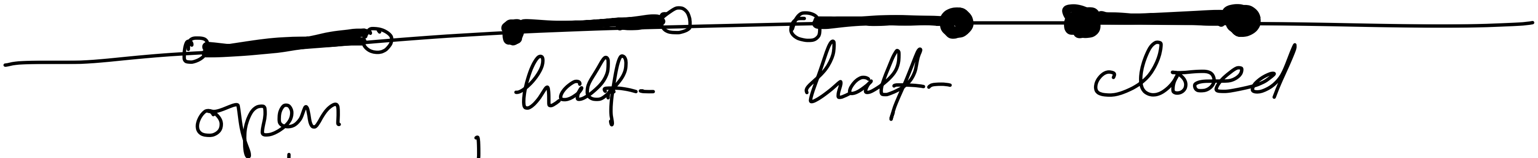
$$\sqrt{\ } : [0, +\infty) \to \mathbb{R}$$

$\lim_{x \to 0} \sqrt{x}$ "does not exist"

$$\lim_{x \downarrow 0} \sqrt{x} = 0$$

To cover this case we modify the definition of continuity, by restricting D to be an interval in $\mathbb{R}$.

finite intervals



infinite intervals

$\mathbb{R} = (-\infty, +\infty)$

9 kinds of intervals.

Definition Let $D \subseteq \mathbb{R}$ be an interval.
A function $f: D \to \mathbb{R}$ is continuous at $c \in D$ if the following condition is satisfied

$$\forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad \text{such that}$$

$\forall x \in D$ we have $|x-c| < \delta(\varepsilon) \Rightarrow |f(x)-f(c)| < \varepsilon$

Def. Let $D \subseteq \mathbb{R}$ be an interval. A function $f: D \to \mathbb{R}$ is CONTINUOUS on D if it is continuous at every $c \in D$.

The complete version of the preceding def. is:

Definition Let $D \subseteq \mathbb{R}$ be an interval. A function $f: D \to \mathbb{R}$ is <u>CONTINUOUS on D</u> if the following condition is satisfied:

$$\forall c \in D \ \forall \varepsilon > 0 \ \exists \delta(\varepsilon, c) > 0 \text{ such that}$$

$$\forall x \in D \quad \boxed{|x - c| < \delta(\varepsilon, c)} \Rightarrow \boxed{|f(x) - f(c)| < \varepsilon}.$$

If $\delta(\varepsilon, c)$ can be chosen such that it does not depend on c, that is $\delta(\varepsilon, c) = \delta(\varepsilon)$, then $f: D \to \mathbb{R}$ is said to be <u>uniformly CONTINUOUS</u>.

Example $f(x) = \frac{1}{x}$ for $x \in (0, +\infty)$

Here $D = (0, +\infty)$. We need to find $\delta(\varepsilon, c) > 0$ for any $c > 0$, $\varepsilon > 0$ such that

$\forall x > 0 \quad |x-c| < \delta(\varepsilon, c) \Rightarrow \left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon$

To discover $\delta(\varepsilon, c)$ we study $\left|\frac{1}{x} - \frac{1}{c}\right|$ and want to tie it to $|x-c|$.

Assume $c > 0$. Remember from limits, we can restrict our x to be close to c ($\delta_0 > 0$) Here $\delta_0 = c/2$

Then restrict x to $\left(\frac{c}{2}, \frac{3c}{2}\right)$

Assume $x \in \left(\frac{c}{2}, \frac{3c}{2}\right) \Longleftrightarrow |x-c| < \frac{c}{2}$

$$\left|\frac{1}{x} - \frac{1}{c}\right| \le \underbrace{??}_{?} \, |x-c|$$