Monotone Convergence Theorem

On Friday we proved:

<u>Thm</u> Every convergent sequence is bounded.

<u>Definition</u> A sequence $s: \mathbb{N} \to \mathbb{R}$ is

nondecreasing if $\forall n \in \mathbb{N}$ $s_n \leq s_{n+1}$

nonincreasing if $\forall n \in \mathbb{N}$ $s_n \geq s_{n+1}$

A sequence with either of these properties is said to be MONOTONIC

Monotone Convergence Theorem

A bounded monotonic sequence converges.

Since monotone has two flavors there are two versions:

① If a sequence is nondecreasing and bdd above, then it converges.

② If a sequence is nonincreasing and bdd below, then it converges.

To prove this Thm we must use the

COMPLETENESS AXIOM.

If A and B are nonempty subsets of $\mathbb{R}$ such that $\forall a \in A$ and $\forall b \in B$ we have $a \leq b$, then $\exists\, c \in \mathbb{R}$ s.t. $\forall a \in A\ \forall b \in B$ $a \leq c \leq b$.

This is the POWER

The set $\mathbb{Q}$ of rational numbers does not satisfy this axiom:

$A = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$

$B = \{y \in \mathbb{Q} : y > 0 \text{ and } y^2 > 2\}$

we can prove $\forall a \in A \ \forall b \in B \quad a < b$,

but $\forall a \in A \ \forall b \in B \quad a \leq c \leq b \Rightarrow c \notin \mathbb{Q}$.

CA is a machine that produces a real number when fed two sets into it.

A & B must be nonempty and A "below" B

$A \subseteq \mathbb{R} \rightarrow \boxed{CA} \rightarrow c \in \mathbb{R}$

$B \subseteq \mathbb{R}$

such that $\forall a \in A \ \forall b \in B$

$a \leq c \leq b$

Proof of MCT.

Assume $s : \mathbb{N} \to \mathbb{R}$ is nondecreasing → English
and bounded above. That is
↳ English

$\forall n \in \mathbb{N} \quad s_n \le s_{n+1}$, that is $s_1 \le s_2 \le \cdots \le s_n \le s_{n+1} \le \cdots$

$\exists M \in \mathbb{R}$ s.t. $\boxed{\forall n \in \mathbb{N} \quad \boxed{s_n \le M}}$

Mathish

In this Green Mathish stuff, do we see some traces of $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$

In green stuff there are some real numbers.

Can I organize them in sets?

It is not useful that A and B are finite sets. Why? Then I can construct $c \in \mathbb{R}$ from other axioms. For example, if $A = \{a\}$ and $B = \{b\}$, $a \le b$ then I can set $c = \frac{a+b}{2}$, then $a \le \underbrace{\frac{a+b}{2}}_{=c} \le b$.

A finite $\max A$ exists. (can be proved)

$\min B$ exists, ...

$\forall a \in A$

$a \le c = \frac{\max A + \min B}{2} \le b \quad \forall b \in B$

Proving is being aware of TOOLS and recognizing tools in GREE STUFF.

The big idea is to **set** (define introduce) the range of s.

$$A = \{s_n : n \in \mathbb{N}\}$$

$$B = \{b \in \mathbb{R} : \forall n \in \mathbb{N} \; s_n \leq b\}$$

Clearly $M \in B$. **So $B \neq \emptyset$.**

Clearly $s_1 \in A$, So $A \neq \emptyset$.