Tests for Convergence of Infinite Series

Direct Comparison Test:

Assume $a_n, b_n > 0$ $\forall n \in \mathbb{N}$ and

$$a_n \leq b_n \quad \forall n \in \mathbb{N}.$$

If $\sum b_n$ converges, then $\sum a_n$ converges.

Limit Comparison Test

Assume $a_n, b_n > 0$ $\forall n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L \quad , \text{ where } L \in \mathbb{R}.$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

More important Test is the

Integral Test:

Let $f : [1, +\infty) \to \mathbb{R}_+$ be a continuous function and decreasing

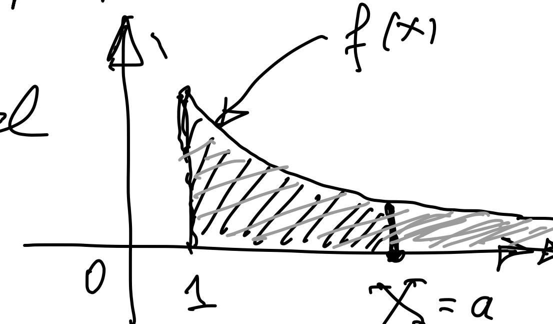
Let $a_n = f(n)\ \forall n \in \mathbb{N}$.

$\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges

Comment about improper integrals:

$\int_1^{+\infty} f(x)\,dx$ is called an improper integral.

$X \geq 1$ $\quad \int_1^X f(x)\,dx$ definite integral



The improper integral $\int_1^{\infty} f(x)\,dx$ is the limit

Math 125 ?

$$\lim_{X\to+\infty} \int_1^{X} f(x)\,dx = \int_1^{+\infty} f(x)\,dx$$

definition

this is just a hidden limit

For us, it will be important to calculat

$\int_1^{\infty} \frac{1}{x^p}\,dx \qquad p\in\mathbb{R} \quad p>1.$

$\int_1^{\infty} \frac{1}{x^{3/2}}\,dx$, first we have calculate $\int_1^{X} \frac{1}{x^{3/2}}\,dx.$

use the Fundamental Theorem of Calculus.

$$\int \frac{1}{x^{3/2}}dx = \int x^{-3/2}dx = -2x^{-1/2}$$

$$(x^{\alpha})' = \alpha x^{\alpha-1}$$

$$-3/2$$

$$\left(-2x^{-1/2}\right)' = x$$

$$\int_1^X \frac{1}{x^{3/2}}dx = \left(-2\frac{1}{\sqrt{x}}\right)\Big|_1^X = -2\frac{1}{\sqrt{X}} + 2$$

Now take $\lim\limits_{X\to+\infty}\left(2 - 2\frac{1}{\sqrt{X}}\right) = 2$

$\to 0$

small for large X

converges to 0

What we proved is $\underbrace{\int_1^\infty \frac{1}{x^{3/2}}\,dx = 2}_{\text{converges}}$

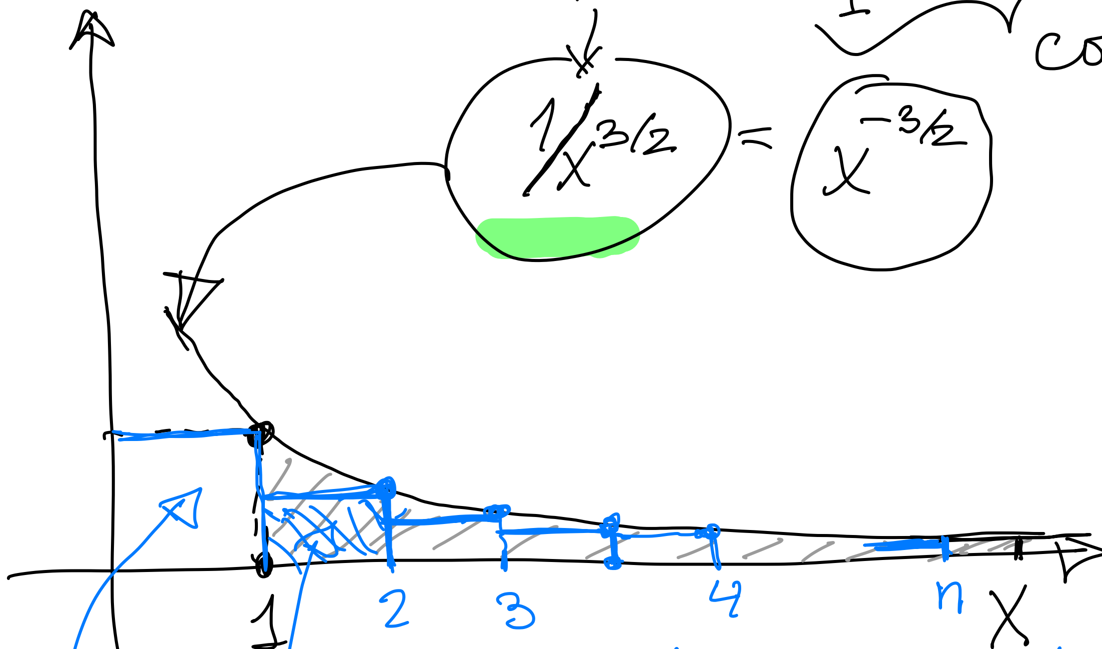
we prefer

$1/x^{3/2} = x^{-3/2}$

$x < x^{3/2} < x^2$

$\frac{1}{x^2} < \frac{1}{x^{3/2}} < \frac{1}{x}$

$= 2$



How is this related to an infinite series?

$$1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \frac{1}{4^{3/2}} + \cdots + \frac{1}{n^{3/2}} < 2$$

$$\sum_{k=1}^{n} \frac{1}{k^{3/2}} < 2$$

bounded by 2

increasing ~~Must~~ must converge!

thus we proved that $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ Converges

We can use the integral test to prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for every $p > 1$.

<0 want >0

$$\int_1^X \frac{1}{x^p}\,dx \overset{FTC}{=} \frac{1}{1-p} x^{1-p}\Big|_1^X = \frac{1}{1-p}\left(X^{1-p} - 1\right) =$$

$$\int \frac{1}{x^p}\,dx = \int x^{-p}\,dx = \frac{1}{1-p} x^{-p+1}$$

anti derivative $\longleftarrow$ derivative

$$\Rightarrow \frac{1}{1-p}\Big(\frac{1}{x^{\underbrace{p-1}_{>0}}} - 1\Big) = \frac{1}{p-1}$$

$$\lim_{x \to +\infty} \frac{1}{x^{\underbrace{p-1}_{>0}}} = 0 \text{ (proof by def)}$$

Thus $\displaystyle\int_1^{+\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1} \rightarrow 0$

$p = 3/2 \dashrightarrow 2$

Therefore, based on the picture,

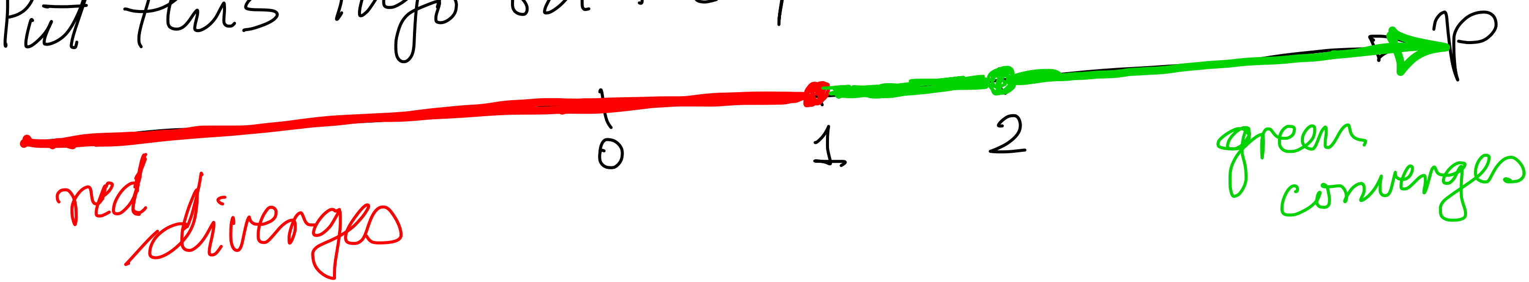
$$\sum_{k=1}^{n} \frac{1}{k^p} < \frac{1}{p-1}$$

Therefore by MCT

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$ converges whenever $p > 1$

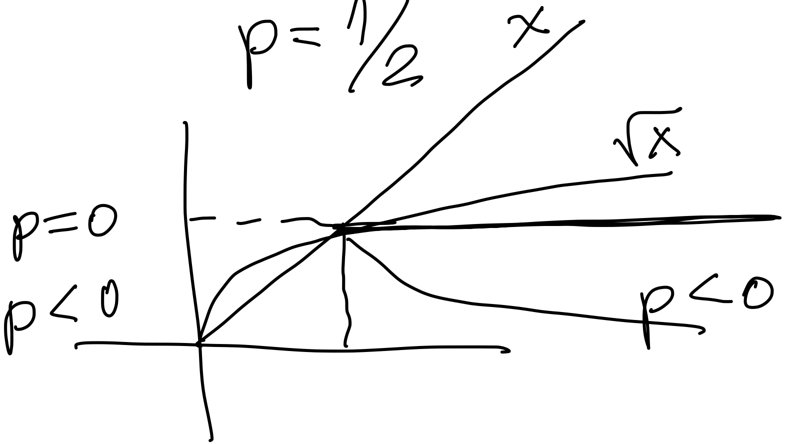
Remember $$\sum_{n=1}^{\infty} \frac{1}{n}$$ diverges.

Put this info on the p-axis:



$p < 1 \quad n \in \mathbb{N} \quad n^p < n$

$p = 1/2$



$$\frac{1}{n^p} > \underbrace{\frac{1}{n}}$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^p}} > \sum \frac{1}{n}$$

diverges as well

diverges

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ this series is called p-series

We know convergence for each one of them.