# Ratio and Root Test for Convergence of Infinite Series

The Geometric Series are <u>central</u> to the Study of Infinite Series.

$a \in \mathbb{R} \setminus \{0\}$, $r \in \mathbb{R} \setminus \{0,1\}$

$$\sum_{n=0}^{+\infty} ar^n$$

$$\sum_{n=0}^{\infty} a_n$$

Is this a Geometric Series?

next → $ar^{n+1}$

previous → $ar^n$

$$\frac{ar^{n+1}}{ar^n} = r$$

<u>constant</u>!

$$\frac{a_{n+1}}{a_n} = ?\ (\text{constant})$$

$r \in (-1,1)$ or $|r|<1$

Converges

$|r| \geq 1$ diverges

**Example** $\displaystyle\sum_{n=0}^{\infty} \frac{(\sqrt{2})^n}{2^{n+1}}$ Looks like G.S.?

How to verify

$$\frac{\text{next} \rightarrow \dfrac{(\sqrt{2})^{\cancel{n+1}}}{2^{\cancel{n+2}}}}{\text{previous} \rightarrow \dfrac{\cancel{(\sqrt{2})^{n}}}{\cancel{2^{n+1}}}} = \frac{\sqrt{2}}{2} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}} < 1$$

This is a G.S. with $r = \dfrac{1}{2\sqrt{2}}$

$a = ?$ $n=0$

$a = \dfrac{1}{2}$

The sum is $a\dfrac{1}{1-r} = \dfrac{1}{2}\dfrac{1}{1-\frac{1}{2\sqrt{2}}}$

simplify $\dfrac{1}{2}\dfrac{2\sqrt{2}}{2\sqrt{2}-1}$

Look at another series:

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{3^n-2^n}}$$ Is this a G.S.?

divide
$\div 3^n$
n&d

Test it $\xrightarrow{\text{next}}$

$$\frac{\dfrac{1}{3^{n+1}-2^{n+1}}}{\dfrac{1}{3^n-2^n}} = \frac{3^n-2^n}{3^{n+1}-2^{n+1}} = \boxed{\frac{1-\left(\frac{2}{3}\right)^n}{3-2\left(\frac{2}{3}\right)^n}} \to \frac{1}{3}$$

$\xrightarrow{\text{previous}}$

use algebra of limits

$\left(\frac{2}{3}\right)^n \to 0$
$(n\to\infty)$

not a constant, but has a limit "CONSTANTISH"

We see $\frac{a_{n+1}}{a_n}$ is NOT constant, but it has a limit $1/3$. In my "slang" not a constant, but "constantish".

## Theorem (Ratio Test)

If $a_n \geq 0$ $\forall n \in \mathbb{N}$,

$\lim\limits_{n \to +\infty} \frac{a_{n+1}}{a_n} = R$, then

If $R < 1$, then $\sum a_n$ converges

If $R > 1$, then $\sum a_n$ diverges

If $R = 1$, can not decide.

· Another way of testing whether a series is geometric or not is by taking n-th root of $a_n$:

$r > 0$ $\sum r^n$, $\sqrt[n]{r^n} = r$

ignore a

If $a_n > 0$, In general to test $\sum a_n$, we take

$$\lim_{n \to +\infty} \sqrt[n]{a_n} = R$$

$R < 1 \Rightarrow \sum a_n$ Converges

$R > 1 \Rightarrow \sum a_n$ Diverges

ROOT TEST

Examples

Ratio Test

Loves factorials

$$\sum_{n=0}^{\infty} \underbrace{\frac{3^n n^2}{n!}}_{a_n}$$ converges or not? $\boxed{>0}$

Ratio test

"next" $\dfrac{\dfrac{3^{n+1}(n+1)^2}{(n+1)!}}{\dfrac{3^n n^2}{n!}}$ "previous"

cancel

cancel

$$= \frac{3(n+1)^2}{n^2(n+1)} = 3\left(\frac{1}{n}+\frac{1}{n^2}\right) \to 0 \quad (n \to \infty)$$

By Ratio Test this series converges.

Example

Let $x \in \mathbb{R}_+$ $\qquad \sum_{n=0}^{+\infty} \frac{x^n}{n!}$ function of $x$

$$\sum_{n=0}^{+\infty} \frac{1}{n!} = e$$

Ratio Test

$$\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1} \to 0$$

Converges!

It TURNS out

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

# Alternating Series

A special kind of series that change sign from positive to negative. The most famous alternating series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

deposit

w.

dep

w.

Alternating Harmonic Series

In general, $a_n > 0$

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

**Theorem** Assume

1. $a_n > 0 \quad \forall n \in \mathbb{N}$
2. $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$
3. $\lim_{n \to +\infty} a_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

**This is AST** Alternating Series Test