Absolute

vs.

Conditional

Convergence of

Infinite Series

Alternating Series Test

If the following three conditions are satisfied:

1. $\forall n \in \mathbb{N} \quad a_n > 0$
2. $\forall n \in \mathbb{N} \quad a_{n+1} \leq a_n$
3. $\lim_{n \to +\infty} a_n = 0$,

then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ CONVERGES

The most importand Example is the Alternating Harmonic Series

$$\underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots}_{= \ln 2}$$

Think of this series as balancing a checkbook with infinite number of transactions.

Total deposits $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ total withdrawals $\sum_{n=1}^{\infty} \frac{1}{2n}$

divergent series !

Total activity in this account is

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$ harmonic series diverges

Since infinite amount is coming in we can start "spending more" earlier:

$$\underbrace{1}_{b_1} - \underbrace{\frac{1}{2}}_{b_2} - \underbrace{\frac{1}{4}}_{b_3} + \underbrace{\frac{1}{3}}_{b_4} - \underbrace{\frac{1}{6}}_{b_5} - \underbrace{\frac{1}{8}}_{b_6} + \underbrace{\frac{1}{5}}_{b_7} - \underbrace{\frac{1}{10}}_{b_8} - \underbrace{\frac{1}{12}}_{b_9} + \dots$$

$$b_{3k-2} = \frac{1}{2k-1}$$

$$b_{3k-1} = \frac{1}{4k-2}$$

$$b_{3k} = \frac{1}{4k}$$

k	$3k-2$	$\frac{1}{2k-1}$
1	1	1/1
2	4	1/3
3	7	1/5
4	11	1/7

$$S_{3n} = \underbrace{1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} \cdots + \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}}_{\text{Finite Sum?}}$$

exactly n deposits

exactly 2n withdrawals

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots + \frac{1}{4n-2} - \frac{1}{4n}$$

$$S_{3n} = \frac{1}{2}\underbrace{\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots+\frac{1}{2n-1}-\frac{1}{2n}\right)}$$

Exactly the 2nth-partial sum of the Alternating Harmonic Series $\longrightarrow \ln 2$

converges to $\ln 2$

$$S_{3n} \longrightarrow \frac{1}{2}\ln 2 \quad (n \to +\infty)$$

S_{3n-1}

S_{3n-2}

alternating harmonic series

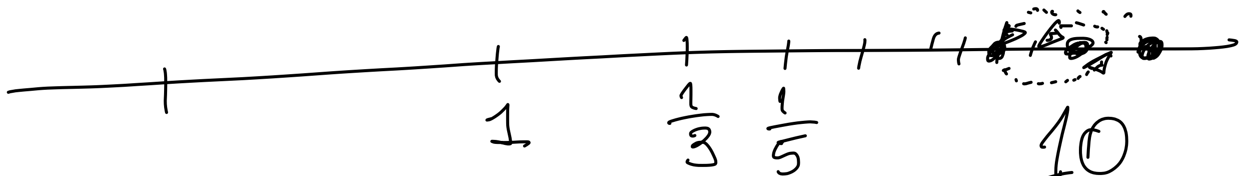
A

An amazing fact is that we can to <u>reorder</u> the terms of the AHS to converge to <u>any number</u>.

How to do that?

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots = +\infty$$

spend



This is called CONDITIONAL CONVERGENCE

Formal Definition: An infinite series $\sum_{n=1}^{\infty} b_n$ is called CONDITIONALLY CONVERGENT if $\sum_{n=1}^{\infty} |b_n|$ diverges.

If $\sum_{n=1}^{\infty} |b_n|$ CONVERGES then the series $\sum_{n=1}^{\infty} b_n$ is called ABSOLUTELY CONVERGENT.

An example of absolutely convergent series is

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots$$

We proved

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^2} \right| = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\text{converges}} = \frac{\pi^2}{6}$$

total activity in the acc.

How much has been withdrawn from this account?

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \underbrace{\frac{1}{4}\frac{\pi^2}{6}}_{\text{total withdra}}$$

total deposited $\pi^2/6$.

thus the balance is $\frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} = \frac{3}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}$

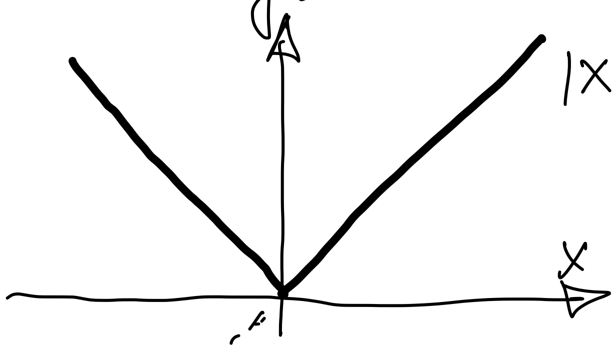
The sum $$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{8}$$

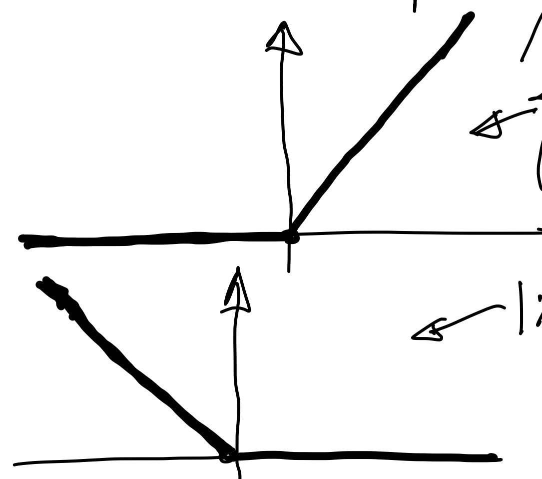
__Theorem__ If a series converges absolutely then it converges.

that is:

$$\underbrace{\sum_{n=1}^{\infty} |b_n| \text{ CONVERGES}}_{\text{absolute convergence}} \Rightarrow \underbrace{\sum_{n=1}^{\infty} b_n \text{ CONVERGES}}_{\text{convergence}}$$

<u>Proof.</u> Absolute Value is an important function.





This is an interplay $|X|$ and X

$$|X|_+ = \frac{1}{2}(|X| + X)$$

$$|X|_- = \frac{1}{2}(|X| - X)$$

$$|X|_+ + |X|_- = |X|$$

$$|X|_+ - |X|_- = X$$

Basicly from $|X|_+$ and $|X|_-$ you can build both X and $|X|$. Also

$$|X|_+ \leq |X| \quad \text{and} \quad |X|_- \leq |X|$$