A squeeze for two common sequences that converge to e

The following two sequences are commonly used to define the number e:

$$S_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad P_n = \left(1 + \frac{1}{n}\right)^n, \qquad n \in \mathbb{N}.$$

Here $\mathbb{N}$ denotes the set of positive integers.

In this note we give a direct proof that $\{S_n\}$ and $\{P_n\}$ converge to the same limit. The main tool in our proof is the Squeeze Theorem, which is probably the easiest to prove among the limit theorems. However, to use the Squeeze Theorem, we need to establish a relevant squeeze, which is the main result of this note.

Surprisingly, many elementary mathematical analysis textbooks do not include a proof that $\{S_n\}$ and $\{P_n\}$ converge to the same limit. The proofs in the classical book [**4**, Theorem 3.31], and in more recent [**1**, Proposition 3.3.1] and [**3**, Appendix 2] all use a limit theorem which they do not prove.

The sequence $\{S_n\}$ converges For completeness we give the standard proof that $\{S_n\}$ is bounded above by 3.

Clearly, $S_1 = 2 < 3$ and as $1/k! \le 1/\big((k-1)k\big)$ for all $k > 1$ we have that

$$S_n = \sum_{k=0}^{n} \frac{1}{k!} \le 2 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 3 - \frac{1}{n},$$

for all $n > 1$. Thus $S_n < 3$ for all $n \in \mathbb{N}$. Since $\{S_n\}$ is increasing, it converges by the Monotone Convergence Theorem.

The squeeze The squeeze that we announced earlier is:

$$S_n - \frac{3}{2n} \le P_n \le S_n \qquad \text{for all} \qquad n \in \mathbb{N}. \tag{1}$$

Applying the Squeeze Theorem to (1) shows that $\{P_n\}$ converges to the same limit as $\{S_n\}$, namely e.

Proof of the squeeze Since (1) is true for $n \le 2$, consider $n > 2$. Our proof of (1) is a succession of four steps, each suggesting the next one.

1. The **binomial theorem** yields an expanded expression for P_n:

$$P_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \, \frac{n!}{(n-k)!\, n^k}. \tag{2}$$

2. For $k \in \{2, \ldots, n\}$, we **rewrite the coefficient** with $1/k!$ in (2) as the product of $k-1$ factors:

$$\frac{n!}{(n-k)!\, n^k} = \frac{(n-1)\cdots(n-k+1)}{n^{k-1}} = \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right). \tag{3}$$

3. An **upper bound** for the product in (3) is clearly 1, so we look for its **lower bound** next. We proceed recursively. At each step, in some sense, we turn a product into a smaller sum.

For $k = 2$ the product in (3) has only one term and obviously $(1 - 1/n) \geq 1 - 1/n$. For $k = 3$, we expand the product and drop a positive term to get:

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) = 1 - \frac{1+2}{n} + \frac{1 \cdot 2}{n} > 1 - \frac{1+2}{n}.$$

For $k = 4$, we multiply both sides above by $\left(1 - \frac{3}{n}\right)$, expend the product on the right and drop a positive term to get:

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) > \left(1 - \frac{1+2}{n}\right)\left(1 - \frac{3}{n}\right) > 1 - \frac{1+2+3}{n}.$$

Repeating this process a total of $k-1$ times and, at the end, using the familiar formula $1 + 2 + \cdots + (k-1) = (k-1)k/2$ (whose history is given in the impressive collection [2]) yields

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{k-1}{n}\right) > 1 - \frac{1 + \cdots + (k-1)}{n} = 1 - \frac{(k-1)k}{2n}.$$

In conclusion, for the product in (3) we have

$$1 - \frac{(k-1)k}{2n} < \frac{n!}{n^k (n-k)!} < 1 \quad \text{for all} \quad k \in \{2, \ldots, n\}. \tag{4}$$

4. The inequalities in (4) are applied to the most right expression in (2) to **establish the inequalities** for P_n:

$$1 + 1 + \sum_{k=2}^{n} \frac{1}{k!}\left(1 - \frac{(k-1)k}{2n}\right) < P_n < 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \cdot 1 = S_n. \tag{5}$$

A simplification of the left-hand side of (5) leads to

$$\sum_{k=0}^{n} \frac{1}{k!} - \sum_{k=2}^{n} \frac{1}{k!} \frac{(k-1)k}{2n} = S_n - \frac{1}{2n} \sum_{k=2}^{n} \frac{1}{(k-2)!} = S_n - \frac{1}{2n} S_{n-2}.$$

Further, since $S_{n-2} < 3$, we have

$$S_n - \frac{1}{2n} S_{n-2} > S_n - \frac{3}{2n}.$$

Consequently, the left-hand side of (5) is greater than $S_n - 3/(2n)$. This proves the squeeze in (1) for all $n > 2$ and completes the proof.

References

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4. Walter Rudin, *Principles of Mathematical Analysis*. Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill, 1976.

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