A BRIEF REVIEW OF MATHEMATICAL LOGIC

BRANKO ĆURGUS

1. Logic

Proofs in mathematics are based on logic. Logic is a science that studies forms of rigorous reasoning. The basic building blocks of logic are propositions.

2. Propositional Calculus

2.1. **Propositions.** A *proposition* or *statement* is a declarative sentence which is either true or false. The following sentence is a proposition: Two plus two equals four. Using the mathematical notation the preceding proposition is written as: $2 + 2 = 4$. This proposition is true. An example of a false proposition is: $0 = 1$.

2.2. **Compound propositions.** Next we introduce natural ways of combining propositions into new ones; these new propositions are called *compound propositions.* In algebra and in other math classes we use letters to stand for numbers. Here we use letters to stand for propositions. The most important compound propositions in mathematics are: the negation, the conjunction, the disjunction, the implication (or conditional) and the equivalence (or biconditional).

Definition 2.1. The *negation* of a proposition p is the proposition "not p" which is false when p is true and which is true when p is false. This proposition is denoted by $\neg p$.

Definition 2.2. The *conjunction* of propositions p and q is the proposition "p and q" which is true when both p and q are true and false otherwise. This proposition is denoted by $p \wedge q$. The conjunction of three propositions p, q and r is defined as $(p \wedge q) \wedge r$ which is true when all three propositions are true and false otherwise.

Definition 2.3. The *disjunction* of propositions p and q is the proposition "p or q" which is false when both p and q are false and true otherwise. This proposition is denoted by $p \vee q$. The disjunction of three propositions p, q and r is defined as $(p \vee q) \vee r$ which is false when all three propositions are false and true otherwise.

Definition 2.4. The *implication* or *conditional* of propositions p and q is the proposition "If p, then q" which is false when p is true and q is false and true otherwise. This proposition is denoted by $p \Rightarrow q$.

Definition 2.5. The *equivalence* or *biconditional* of propositions p and q is the proposition "p if and only if q" which is true when both $p \Rightarrow q$ and $q \Rightarrow p$ are true. This proposition is denoted by $p \Leftrightarrow q$.

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The above definitions are summarized by the following *truth tables*.

negation

p	$\neg p$
T	F
F	T

conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

implication

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

equivalence

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

2.3. **Propositional calculus.** Doing calculations with propositions is called *propositional calculus*. Here are the most important rules of propositional calculus.

The double negation rule: *The negation of the negation of a proposition is equivalent to the original proposition.* In short: $\neg(\neg p) \Leftrightarrow p$. To prove this proposition we form the truth table:

p	$\neg p$	$\neg(\neg p)$	p
T	F	T	T
F	T	F	F

DeMorgan's Laws: *The negation of a conjunction is the disjunction of the corresponding negations.* In short: $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$. The proof, again, is by truth table:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$(\neg p) \vee (\neg q)$	$\neg p$	$\neg q$
T	T	T	F	F	F	F
T	F	F	T	T	F	T
F	T	F	T	T	T	F
F	F	F	T	T	T	T

The negation of a disjunction is the conjunction of the corresponding negations. In short: $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$. The proof, again, is by truth table:

p	q	$p \vee q$	$\neg(p \vee q)$	$(\neg p) \wedge (\neg q)$	$\neg p$	$\neg q$
T	T	T	F	F	F	F
T	F	T	F	F	F	T
F	T	T	F	F	T	F
F	F	F	T	T	T	T

The negation of an implication: *The negation of* $p \Rightarrow q$ *is the conjunction of* p *and the negation of* q. In short: $\neg(p \Rightarrow q) \Leftrightarrow \big(p \wedge (\neg q)\big)$. The proof, again, is by truth table:

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$p \wedge (\neg q)$	p	$\neg q$
T	T	T	F	F	T	F
T	F	F	T	T	T	T
F	T	T	F	F	F	F
F	F	T	F	F	F	T

3. Implications

Since the most important statements in mathematics are formulated as implications, the implications deserve a section in this brief summary.

Probably because of its importance, the English language has provided twelve different ways of verbalizing the implication $p \Rightarrow q$: $\boxed{\text{(I) If } p\text{, then } q.}$ $\boxed{\text{(II) If } p, q.}$ $\boxed{\text{(III) } q \text{ if } p.}$

(IV) q when p.	(V) p is sufficient for q.	(VI) q is necessary for p.	
(VII) A sufficient condition for q is p.	(VIII) A necessary condition for p is q.		
(IX) p implies q.	(X) p only if q.	(XI) q whenever p.	(XII) q follows from p.

For a given implication, there are three named related implications.

Definition 3.1. Let $p \Rightarrow q$ be an implication. The *contrapositive* of $p \Rightarrow q$ is the implication $\neg q \Rightarrow \neg p$. The *converse* of $p \Rightarrow q$ is the implication $q \Rightarrow p$. The *inverse* of $p \Rightarrow q$ is the implication $\neg p \Rightarrow \neg q$.

Although we are interested in mathematical propositions, it is often useful to explore real life propositions. My favorite real-life example relates to Red Square, the main square with a fountain on the WWU campus.

Name	Claim
p	It rains on Red Square.
q	Red Square is wet.
$\neg p$	It does not rain on Red Square.
$\neg q$	Red Square is not wet.

When dealing with real-life we assume that the claims that we use as propositions truly are either true or false. We do not accept that there are ambiguous situation in which we can not decide whether it is raining or not.

Let us explore the implication $p \Rightarrow q$ and three related implications which involve the above real-life propositions p and q.

	Claim	Name
$p \Rightarrow q$	If it rains on Red Square, then Red Square is wet.	Implication
$q \Rightarrow p$	If Red Square is wet, then it rains on Red Square.	Converse
$\neg q \Rightarrow \neg p$	If Red Square is not wet, then it does not rain on Red Square.	Contrapositive
$\neg p \Rightarrow \neg q$	If it does not rain on Red Square, then Red Square is not wet.	Inverse

In my experience the implication $p \Rightarrow q$ in the preceding table is true. I have never witnessed that Red Square is not wet when it is raining on Red Square. In my experience the converse $q \Rightarrow p$ is not always true. On few occasions, I have witnessed wet Red Square, on a sunny day. For example, the fountain flooded once and Red Square was truly wet on a sunny, that is a definitely nonrainy, day. From this we conclude that the implication and its converse are not related. It is possible for an implication to be true, while the converse is not true. In fact, whenever you see an implication proved in mathematics, a good question to ask: Is the converse true?

The situation with the contrapositive is different: *A proposition is equivalent to its contrapositive.* The easiest way to see this is to look at the negations. The negation of the implication $p \Rightarrow q$ is $p \wedge (\neg q)$. The negation of the contrapositive $(\neg q) \Rightarrow (\neg p)$ is $(\neg q) \wedge \big(\neg(\neg p)\big)$.

By the double negation rule the proposition $\neg(\neg p)$ is equivalent to p. Hence the negation of the contrapositive is equivalent to $(\neg q) \wedge p$. But, $p \wedge (\neg q)$ and $(\neg q) \wedge p$ are clearly equivalent. So, the negations of the implication and its contrapositive have the identical truth values. Therefore the implication and its contrapositive have the identical truth values.

The fact that *a proposition is equivalent to its contrapositive* can also be seen by forming the following truth table:

p	q	$p \Rightarrow q$	$(\neg q) \Rightarrow (\neg p)$	$\neg q$	$\neg p$
T	T	T	T	F	F
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Definition 3.2. Let $p \Rightarrow q$ be an implication. The proposition p is called the *hypothesis* of the implication $p \Rightarrow q$. The proposition q is called the *conclusion* of the implication $p \Rightarrow q$.

4. Propositional functions

4.1. **Propositional functions.** The next level in the hierarchy of logical constructs are propositional functions. A *propositional function* or *predicate* is an expression which involves one or more variables and which becomes a proposition when variables are replaced by specific values from a particular set of values called the *universe of discourse*.

For example: $2x^2 - x > 0$ is not a proposition; it is a propositional function whose universe of discourse can be any set of numbers. Denote the the propositional function $2x^2 - x > 0$ by $Q(x)$. Let $S = \{-1, 0, 1\}$ be the universe of discourse for $Q(x)$. Then we can form three specific propositions $Q(-1), Q(0), Q(1)$:

x	$Q(x)$	meaning	T or F
-1	$Q(-1)$	$3 > 0$	T
0	$Q(0)$	$0 > 0$	F
1	$Q(1)$	$1 > 0$	T

Now we can form the conjunction of these three propositions $Q(-1) \wedge Q(0) \wedge Q(1)$ which is

$$(3 > 0) \wedge (0 > 0) \wedge (1 > 0)$$

which is false since not all of the three propositions are true. The negation of this compound proposition

$$(3 \leq 0) \vee (0 \leq 0) \vee (1 \leq 0)$$

is true since the proposition $0 \leq 0$ is true.

Similarly we can form the disjunction of the propositions in the table: $Q(-1) \vee Q(0) \vee Q(1)$ which is

$$(3 > 0) \vee (0 > 0) \vee (1 > 0)$$

which is true since not all of the three propositions are false. The negation of the preceding compound proposition

$$(3 \leq 0) \wedge (0 \leq 0) \wedge (1 \leq 0)$$

is false since not all three propositions in the conjunction are true; the proposition $1 \leq 0$ is false.

4.2. **Quantifiers.** If we consider the universe of discourse of $Q(x)$ to be the set $\mathbb{R}$ of all real numbers, then forming the conjunction and the disjunction of all the propositions $Q(x)$ with $x \in \mathbb{R}$ would be impossible in a way that we did above. Therefore we introduce the concept of the *universal quantifier* and the *existential quantifier*.

The symbol for the *universal quantifier* is $\forall$. We read it "for all." If $P(x)$ is any propositional function with the universe of discourse U then

$$\forall x \in U\ P(x) \quad \text{means that} \quad P(x) \text{ is true for all } x \in U.$$

Notice that with $Q(x)$ being $2x^2 - x > 0$ the statement $\forall x \in \mathbb{R}\ Q(x)$ is a proposition whose truth value is F (false) since $2x^2 - x > 0$ is not true for all $x \in \mathbb{R}$. For example, with $x = 0$ we have $2(0)^2 - 0 = 0 > 0$, which is false.

The symbol for the *existential quantifier* is $\exists$. We read it "there exists." If $P(x)$ is any propositional function with the universe of discourse U then

$$\exists x \in U\ P(x) \quad \text{means that} \quad P(x) \text{ is true for at least one } x \in U.$$

or,

$$\exists x \in U\ P(x) \quad \text{means that} \quad \text{there exists } x \in U \ \text{ such that } \ P(x) \text{ is true.}$$

Notice that with $Q(x)$ being $2x^2 - x > 0$ the statement $\exists x \in \mathbb{R}\ Q(x)$ is a proposition whose truth value is T (true) since for $x = 1 \in \mathbb{R}$ we have $2(1)^2 - 1 = 1 > 0$, which is true.

4.3. **Negations of statements with quantifiers.** The meaning of the statements with quantifiers and their negations is best summarized in the following table:

Statement	**When true?**	**When false?**	**Negation**
$\forall x \in U\ P(x)$	$P(x)$ is true for all $x \in U$	There exists an $x \in U$ such that $P(x)$ is false	$\exists x \in U\ \neg P(x)$
$\exists x \in U\ P(x)$	There exists an $x \in U$ such that $P(x)$ is true	$P(x)$ is false for all $x \in U$	$\forall x \in U\ \neg P(x)$

4.4. **Statements with multiple quantifiers.** To make things even more interesting mathematical statements often come with two or more quantifiers. Consider the statement

$$\forall a \in \mathbb{R}\ \ \exists x \in \mathbb{R}\ \ ax^2 - x > 0. \tag{4.1}$$

Is this statement true or false? Its negation is

$$\exists a \in \mathbb{R}\ \ \forall x \in \mathbb{R}\ \ ax^2 - x \leq 0.$$

The statement in (4.1) is true. We can prove it by considering two cases. If $a \geq 0$, then we can take $x = -1$. We get $a(-1)^2 - (-1) = a + 1 \geq 1 > 0$, which is true. If $a < 0$, then take $x = 1/(2a)$. Then, since $a < 0$, we have

$$a\left(\frac{1}{2a}\right)^2 - \frac{1}{2a} = \frac{1}{4a} - \frac{1}{2a} = -\frac{1}{4a} > 0.$$

So, in both cases the statement in (4.1) is true.

It is very important to realize that the order of quantifiers matters. The reversal of the quantifiers in (4.1) leads to the following statement

$$\exists x \in \mathbb{R}\ \forall a \in \mathbb{R}\ \ ax^2 - x > 0.$$

This statement is false since its negation

$$\forall x \in \mathbb{R}\ \exists a \in \mathbb{R}\ \ ax^2 - x \leq 0$$

is true. To prove the negation we consider two cases. The first case $x \geq 0$. Setting $a = 0$ we have $0(x)^2 - x = -x \leq 0$ which is true. The second case is $x < 0$. Assume $x < 0$. Setting $a = 2/x$ we have

$$\frac{2}{x}x^2 - x = x \leq 0,$$

which is a true statement since $x < 0$ in this case.

The simplest unsolved problem in mathematics is Goldbach's Conjecture which states a natural relationship between the even positive integers greater rhan 2 and the primes. The common notation for the set of all positive integers is $\mathbb{N}$ and by $\mathbb{P}$ we denote the set of all primes. Using the notation of mathematical logic Goldbach's Conjecture can be stated as follows:

$$\forall k \in \mathbb{N}\backslash\{1\}\ \ \exists p \in \mathbb{P}\ \ \exists q \in \mathbb{P} \quad 2k = p + q.$$

Using sophisticated programming this proposition has been verified for all $k \in \mathbb{N}\backslash\{1\}$ such that than $k \leq 2 \times 10^{18}$. However, despite significant effort of generations of mathematicians there is no proof that it holds for <u>all</u> integers greater than 1.

5. Examples

In the proof of Proposition 5.1 below I will demonstrate how I use colors in proofs. Most mathematical statements are implications: $p \Rightarrow q$. To prove such a statement we assume p and use some previous mathematical knowledge and logic to deduce q. To emphasise what is **assumed** and what previous mathematical **knowledge** is used, these statements are colored **green**. What needs to be proved, that is q, is colored **red**. As proof progresses we have more and more **green** stuff. Finally, at the end of the proof, the statement q becomes q, that is it is being **greenified**.

Another way that I use colors is by coloring numbers. For example, in your previous mathematical experience you have learned how to solve the quadratic equation: Let $a, b, c \in \mathbb{R}$ and assume that $a \neq 0$. Find $x \in \mathbb{R}$ such that

$$ax^2 + bx + c = 0.$$

Here $a, b, c \in \mathbb{R}$ are assumed to be known numbers (called coefficients) and x is unknown. To emphasise the dichotomy **known**–**unknown**, $a, b, c \in \mathbb{R}$ are colored **green** and $x \in \mathbb{R}$ is colored **red**:

$$ax^2 + bx + c = 0.$$

With this coloring solving the equation becomes just separating the colors. That is, expressing **red** in terms of **green**:

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Using the notation of mathematical logic this short review of quadratic equation can be summarized as follows: Let $a, b, c, x \in \mathbb{R}$ and assume that $a \neq 0$. Then

$$ax^2 + bx + c = 0 \quad \Leftrightarrow \quad \left(x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) \bigvee \left(x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right).$$

The following proposition states something that you might have learned in your previous mathematical experience. Here I state the proposition using the notation of mathematical logic and use colors to make the proof easier to follow.

Proposition 5.1. *For all $a, b, c \in \mathbb{R}$ the following implication holds*

$$\forall x \in \mathbb{R} \quad ax^2 + bx + c \geq 0 \quad \Rightarrow \quad a \geq 0.$$

Proof. Let $a, b, c \in \mathbb{R}$ be arbitrary. The colored version of the implication in the proposition is

$$\boxed{\forall x \in \mathbb{R} \quad ax^2 + bx + c \geq 0} \quad \Rightarrow \quad \boxed{a \geq 0}. \tag{5.1}$$

The implication in (5.1) is equivalent to its contrapositive

$$\boxed{a < 0} \quad \Rightarrow \quad \boxed{\exists x \in \mathbb{R} \quad ax^2 + bx + c < 0}. \tag{5.2}$$

To prove the implication in (5.2) we need to understand the coloring of the variables involved. The real numbers $a, b, c \in \mathbb{R}$ are given. They are arbitrary but fixed real numbers. To emphasise that fact I color them green: $a, b, c \in \mathbb{R}$. The variable $x \in \mathbb{R}$ which is under the existential quantifier is red since we have to find a real number with this specific property:

$$\boxed{\exists x \in \mathbb{R} \quad \text{such that} \quad ax^2 + bx + c < 0}.$$

In fact the redness of $x \in \mathbb{R}$ is the core of the redness of the conclusion in the implication in (5.2). These colors indicate how to prove (5.2): We need to find a formula for a red number in terms of the green numbers. To be more specific: For the given $a, b, c \in \mathbb{R}$ of which you know only that $\boxed{a < 0}$ our task is to find a formula for $x \in \mathbb{R}$ in terms of $a, b, c \in \mathbb{R}$ for which we will be able to prove

$$ax^2 + bx + c < 0.$$

This is the shortest proof that I found. It is somewhat cryptic. I want you to understand that this is a correct proof. I will try to explain how I found it in a comment box.

Since the numbers $b, c \in \mathbb{R}$ are green, the following number is also green

$$d = \max\{b, c, 0\}.$$

In words, d is the largest of the three numbers b, c, and 0. In particular we have the following inequalities

$$\boxed{b \leq d}, \quad \boxed{c \leq d}, \quad \text{and} \quad \boxed{0 \leq d}. \tag{5.3}$$

Set

$$x = x_0 = 1 - \frac{d}{a}.$$

Now calculate

$$\begin{aligned}
a(x_0)^2 + bx_0 + c &= a\left(1 - \frac{d}{a}\right)^2 + b\left(1 - \frac{d}{a}\right) + c \\
&= a\left(1 - 2\frac{d}{a} + \frac{d^2}{a^2}\right) + b - \frac{bd}{a} + c \\
&= a - 2d + \frac{d^2}{a} + b - \frac{bd}{a} + c \\
&= a + (b - d) + (c - d) + \frac{d(d - b)}{a}.
\end{aligned}$$

Now consider the four summands in the preceding sum. We have that $\boxed{a<0}$ by assumption. We have that $\boxed{b-d\leq 0}$, since $\boxed{b\leq d}$ by (5.3). We have that $\boxed{c-d\leq 0}$, since $\boxed{c\leq d}$ by (5.3). We have that $\boxed{d(d-b)\geq 0}$, since $\boxed{d\geq 0}$ and $\boxed{d\geq b}$ by (5.3). Since $\boxed{a<0}$ we have $\boxed{d(d-b)/a\leq 0}$. To summarize,

$$\boxed{a<0}, \quad \boxed{b-d\leq 0}, \quad \boxed{c-d\leq 0}, \quad \boxed{\frac{d(d-b)}{a}\leq 0}.$$

Therefore

$$\boxed{a(x_0)^2+bx_0+c = a+(b-d)+(c-d)+\frac{d(d-b)}{a} < 0}.$$

Since

$$x_0 = 1-\frac{\max\{b,c,0\}}{a}$$

is given in terms of a, b, and c, and

$$\boxed{a(x_0)^2+bx_0+c<0}$$

the proposition is proved. □

Thinking that led to the above proof.

I first decided to study a simpler version of the quadratic expression with $a=-1$. Can I find a simple value of x for which

$$-x^2+bx+c<0?$$

That turned out to be hard. So, I decided to simplify further by making b and c equal and I called it d. Can I find a simple value of x for which

$$-x^2+dx+d<0?$$

This turned out to be simple. I discovered that for $x=1+d$ we have

$$-(1+d)^2+d(1+d)+d = -1-2d-d^2+d+d^2+d = -1.$$

This was great. How to connect it to $-x^2+bx+c$? If I take $b\leq d$ and $c\leq d$, then I will have

$$-x^2+bx+c \leq -x^2+dx+d \quad \text{provided that } x \text{ is positive.}$$

Since I need a positive x and we used $x=1+d$, if we choose a positive d this could all work out. This inspired me to take

$$d=\max\{b,c,0\}.$$

Then

$$-(1+d)^2+b(1+d)+c = -1-2d-d^2+b+bd+c = -1+(b-d)+(c-d)+d(b-d) \leq -1.$$

How to incorporate a into this? I considered $a=-1$ above. Where is -1 in the above formulas? It turned out that $x=1-d/(-1)$ is the right answer. So, that is where $x=1-d/a$ comes from.

The proof of Proposition 5.1 that I wrote above is long. I incorporated some thinking that could have been skipped. Below I present a shorter version of the proof with the just the essentials and without colors.

Proof. Let $a, b, c \in \mathbb{R}$ be arbitrary. We will prove the contrapositive of the implication in the proposition. Set

$$d = \max\big\{b, c, 0\big\}.$$

As a consequence of this definition, the following inequalities hold

$$b \leq d, \quad c \leq d, \quad \text{and} \quad 0 \leq d. \tag{5.4}$$

Next we can define a real number x_0 at which the quadratic expression takes a negative value:

$$x_0 = 1 - \frac{d}{a}.$$

Now calculate

$$a(x_0)^2 + bx_0 + c = a\left(1 - 2\frac{d}{a} + \frac{d^2}{a^2}\right) + b - \frac{bd}{a} + c = a + (b-d) + (c-d) + \frac{d(d-b)}{a}.$$

By assumption we have $a < 0$ and from (5.3) we have $b - d \leq 0$, $c - d \leq 0$, and $d(d-b) \geq 0$. Therefore

$$a(x_0)^2 + bx_0 + c = a + (b-d) + (c-d) + \frac{d(d-b)}{a} < 0.$$

The proposition is proved. $\square$

The preceding proposition is a part of the proof of the first equivalence in the following theorem which states five equivalences related to quadratic functions. If you see more relevant equivalences to add to the theorem below, please let me know.

Theorem 5.2. *Let $a, b, c \in \mathbb{R}$. Then the following equivalences hold*

$$\forall x \in \mathbb{R} \;\; ax^2 + bx + c \geq 0 \quad \Leftrightarrow \quad a \geq 0 \;\wedge\; c \geq 0 \;\wedge\; b^2 - 4ac \leq 0, \tag{5.5}$$

$$\forall x \in \mathbb{R} \;\; ax^2 + bx + c \leq 0 \quad \Leftrightarrow \quad a \leq 0 \;\wedge\; c \leq 0 \;\wedge\; b^2 - 4ac \leq 0, \tag{5.6}$$

$$\forall x \in \mathbb{R} \;\; ax^2 + bx + c > 0 \quad \Leftrightarrow \quad \big(a > 0 \wedge b^2 - 4ac < 0\big) \vee \big(a = 0 \wedge b = 0 \wedge c > 0\big), \tag{5.7}$$

$$\forall x \in \mathbb{R} \;\; ax^2 + bx + c < 0 \quad \Leftrightarrow \quad \big(a < 0 \wedge b^2 - 4ac < 0\big) \vee \big(a = 0 \wedge b = 0 \wedge c < 0\big), \tag{5.8}$$

$$\exists s \in \mathbb{R}\; \exists t \in \mathbb{R} \quad \big(as^2 + bs + c < 0\big) \wedge \big(at^2 + bt + c > 0\big) \quad \Leftrightarrow \quad b^2 - 4ac > 0. \tag{5.9}$$