MATH 302 Examination 1 April 28, 2009

Name Key

Problem 1. (A) Give the definition of $a|b$.

(B) For what integers a is $1|a$ true? Give all such integers a. Prove your claim.

(C) For what integers a is $a|0$ true? Give all such integers a. Prove your claim.

(D) For what integers a is $a|b$ true for all integers b? Give all such integers a. Prove your claim.

Problem 2. Let a be an integer and let n be a positive integer. Prove that the set

$$S = \{x \in \mathbb{Z} : n \mid x \text{ and } x \le a\}$$

has a maximum.

Proposition 3. Let a be an integer and let n be a positive integer. Then there exist unique integers q and r such that

$$a = nq + r \quad \text{and} \quad 0 \le r \le n - 1 \, .$$

Problem 4. If a and b are odd perfect squares, then $a + b$ is not a perfect square.

1A $a|b$ if $a \neq 0$ and $\exists k \in \mathbb{Z}$ $\boxed{1}$
s.t. $b = ak$.

1B $1|a$ is true for all integers a,
since $1 \neq 0$ and
$a = a \cdot 1$ for all $a \in \mathbb{Z}$

1C $a|0$ is true for all $a \in \mathbb{Z} \setminus \{0\}$.
If $a \in \mathbb{Z} \setminus \{0\}$, then $a \neq 0$
and $0 = 0 \cdot a$ for all $a \in \mathbb{Z} \setminus \{0\}$

1D $a|b$ is true for $a = 1$ and all $b \in \mathbb{Z}$
as seen in 1B. But
$(-1)|b$ for all $b \in \mathbb{Z}$ as well.
$a|b$ is not true if $a \notin \{1, -1\}$.
If $a|b$ for all $b \in \mathbb{Z}$, then $a|1$.
$a|1 \Rightarrow a = 1$ or $a = -1$.

(2) $S = \{x \in \mathbb{Z} : n \mid x \ \& \ x \leq a\}$ $\boxed{2}$

$n \in \mathbb{N} \quad a \in \mathbb{Z}$.

Clearly S is bounded above by a.

Is $S = \phi$? No. ~~Consider~~

Clearly $-|a| \in \mathbb{Z}$ and

$-|a| \leq -1$. Therfore (multiply $n \geq 1$ by $-|a| < 0$)

$$-n|a| \leq -|a| \leq a$$

Thus $-n|a| \leq a$ and clearly $n \mid (-n|a|)$.

Thus $-n|a| \in S$, so $S \neq \phi$.

By a proposition proved in class $\max S$ exists.

(3) By Pr. 2 max S exists. $\boxed{3}$

Set $b = \max S$. Since $b \in S$ $n | b$ and $b \le a$.

Therefore $\exists q \in \mathbb{Z}$ such that $b = nq$ and $nq \le a$.

Set $r = a - nq$. Then $\boxed{a = nq + r}$ and $r \geq 0$. We need to prove that $r < n$. Since $b = \max S$ $b + n = n(q+1) \notin S$. ~~Theref~~

Since $n | n(q+1)$ we conclude $nq + n > a$

Hence $nq + n > nq + r$. Therefore $n > r$.

This prove the existence of q and r.

To prove uniqueness assume

$$a = nq + r \qquad 0 \le r < n$$

$$a = nq' + r' \qquad 0 \le r' < n$$

$$r - r' = a - nq - a + nq', \quad -n < r - r' < n$$

$$= n(q' - q)$$

So $-n < n(q' - q) < n$

So $-1 < q' - q < 1$ so $q' = q$.

and consequently $r - r' = n(q' - q) = 0$, so $r' = r$.

(4) Assume a and b are odd perfect squares, that is

$$a, b \in \mathbb{O} \cap \mathbb{S}$$

Since $a, b \in \mathbb{O}$, as we proved in class $a+b \in \mathbb{E}$. Since ~~$a = 2k+1$ and~~

Let $a = u^2$ and $b = v^2$. We proved in class that $u, v \in \mathbb{O}$. Hence $u = 2x+1$ and $v = 2y+1$, $x, y \in \mathbb{Z}$. Thus $a = 4x^2+4x+1$ and $b = 4y^2+4y+1$. Hence $a+b = 4(x^2+x+y^2+y)+2$. Thus remainder when $a+b$ is divided by 4 is 2, that is $a+b \in {}_4\mathbb{R}_2$.

We proved in class that even squares have remainder 0 when divided by 4. That is

$$c \in \mathbb{E} \cap \mathbb{S} \Rightarrow 4 | c$$

The contrapositive is

$$4 \nmid c \Rightarrow c \notin \mathbb{E} \cap \mathbb{S}$$

Clearly $4 \nmid (a+b)$ so $a+b \notin \mathbb{E} \cap \mathbb{S}$

Since $a+b \in \mathbb{E}$, we conclude $a+b \notin \mathbb{S}$