MATH 304 Final Examination
August 19, 2010

Name Key

GIVE DETAILED EXPLANATIONS FOR YOUR ANSWERS.
There are five problems. The best four count for 25 points each.

1. Let A be a 2×2 matrix with the eigenvalues $\lambda_1 = 2, \lambda_2 = 1/2$, and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Let $\mathbf{x}_0 = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and define $\mathbf{x}_k = A^k \mathbf{x}_0$ for $k = 1, 2, 3, \ldots$.

   (a) Express $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Use this fact to compute $\mathbf{x}_k$. (Hint: Note that the entries of A are unknown and, in fact, they are not needed!)

   (b) Describe what happens to $\mathbf{x}_k$ as $k \to \infty$. Be specific, describe how both the norm and the direction of $\mathbf{x}_k$ change as $k \to \infty$.

2. Find a QR-factorization of the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

3. Let A be a symmetric matrix. Prove that all the eigenvalues of A are real.

4. Consider the vector space $\mathbb{P}_2$ with the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)dt.$$

   Let

$$p_0(t) = 1, \qquad p_1(t) = t, \qquad p_2(t) = t^2.$$

   be the standard basis for $\mathbb{P}_2$.

   (a) Calculate $\langle p_0, p_0 \rangle$, $\langle p_0, p_1 \rangle$, $\langle p_0, p_2 \rangle$, $\langle p_1, p_1 \rangle$, $\langle p_1, p_2 \rangle$.

   (b) Compute the orthogonal projection of p_2 onto the subspace spanned by p_0 and p_1.

   (c) Find an orthogonal basis for this inner product space. Denote the polynomials in this orthogonal basis by q_0, q_1 and q_2. Normalize these polynomials so that $q_0(1) = 1, q_1(1) = 1$ and $q_2(1) = 1$.

5. (a) For the symmetric matrix $S = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$ find a diagonal matrix D and an orthogonal matrix P such that $S = PDP^T$.

   (b) Find a singular value decomposition of the matrix $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$. Notice that $A^T A = S$.

   (c) Find a unit vector $\mathbf{x}$ at which $A\mathbf{x}$ has maximum length. Find the vector $A\mathbf{x}$ and its length.

$\boxed{1}$

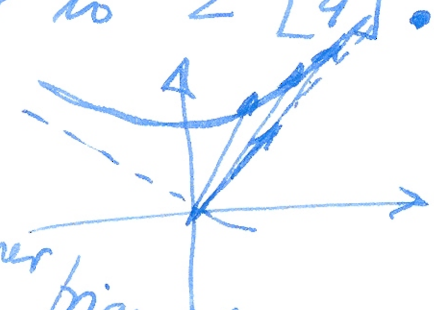
1. ⓐ $\begin{bmatrix}7\\1\end{bmatrix} = 4\begin{bmatrix}1\\1\end{bmatrix} - 3\begin{bmatrix}-1\\1\end{bmatrix}$

$$\vec{x}_k = A^k\begin{bmatrix}7\\1\end{bmatrix} = 4\cdot 2^k\begin{bmatrix}1\\1\end{bmatrix} - 3\left(\frac{1}{2}\right)^k\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= 2^k\begin{bmatrix}4\\4\end{bmatrix} - \left(\frac{1}{2}\right)^k\begin{bmatrix}-3\\3\end{bmatrix}$$

ⓑ $\vec{x}_k = 2^k\begin{bmatrix}4\\4\end{bmatrix} - \left(\frac{1}{2}\right)^k\begin{bmatrix}-3\\3\end{bmatrix}$

Since $\left(\frac{1}{2}\right)^k \to 0$ as $k \to +\infty$, we conclude that $\vec{x}_k$ becomes longer and longer and closer and closer to $2^k\begin{bmatrix}4\\4\end{bmatrix}$.



2. We need $A = QR$

Q: orthonormal columns; R: upper triangular

$\vec{a}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$

$$\text{Proj}_{\vec{a}_1}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}; \quad \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}$$

$$\begin{bmatrix}-1\\1\\2\end{bmatrix} \perp \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad \begin{bmatrix}-1\\1\\2\end{bmatrix} = -\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$Q = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{2} & 1/\sqrt{6}\\ 0 & 2/\sqrt{6}\end{bmatrix} \qquad R = Q^TA = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\sqrt{2} & \sqrt{2}/2\\ 0 & \sqrt{6}/2\end{bmatrix}$$

$\cancel{A = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}}$ $\boxed{2}$

orthonormal columns　　　　upper triangular

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$ Yes!

3. $\vec{v} \neq \vec{0}$ and $A\vec{v} = \lambda \vec{v}$

$$A^T = A$$

I will write vectors without an arrow, and $\bar{v}$ means a complex conjugate.

Since A is a real matrix $\bar{A} = A$.

Since $\bar{v} = \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix}$, ~~we have~~ and $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \neq 0$

we have $\bar{v}^T v = |v_1|^2 + \dots + |v_n|^2 > 0$

not all ~~...~~ components are 0.

Now we calculate $\bar{v}^T A v$ in two different ways: $\boxed{3}$

$$\bar{v}^T A v = \bar{v}^T(\lambda v) = \lambda \bar{v}^T v = \lambda(|v_1|^2 + \dots + |v_n|^2)$$

$$\bar{v}^T A v = \bar{v}^T A^T v = (A\bar{v})^T v = (\bar{\lambda}\bar{v})^T v = \bar{\lambda}\bar{v}^T v = \bar{\lambda}(|v_1|^2 + \dots + |v_n|^2)$$

(Here we used that $Av = \lambda v$ and $A\bar{v} = \bar{\lambda}\bar{v}$ and $A^T = A$.)

Hence

$$\lambda \underbrace{(|v_1|^2 + \dots + |v_n|^2)}_{>0} = \bar{\lambda}\underbrace{(|v_1|^2 + \dots + |v_n|^2)}_{>0}$$

So $\lambda = \bar{\lambda}$ is a real number.

4 (a) $\boxed{4}$

$\langle p_0, p_0 \rangle = 2$

$\langle p_0, p_1 \rangle = 0$ p_0, p_1 orthogonal ☺

$\langle p_0, p_2 \rangle = \int_{-1}^{1} t^2 dt = \frac{2}{3}$

$\langle p_1, p_1 \rangle = \int_{-1}^{1} t \cdot t \, dt = 2/3$

$\langle p_1, p_2 \rangle = \int_{-1}^{1} t^3 dt = 0$

(b)

$$\mathrm{Proj}_{\mathrm{span}\{p_0, p_1\}} p_2 = \frac{\langle p_2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 + \frac{\langle p_2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1$$

$$= \frac{2/3}{2} \cdot p_0 + \frac{0}{2/3} p_1$$

$$= \frac{1}{3} p_0$$

(c) An orthogonal basis is

$$p_0, p_1, p_2 - \frac{1}{3} p_0$$

$q_0 = p_0$ since $p_0(1) = 1$

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$q_0 = p_0$, since $p_0(1) = 1$

$q_1 = p_1$ since $p_1(1) = 1$

$$p_2(1) - \frac{1}{3}p_0(1) = 1 - \frac{1}{3} = \frac{2}{3}$$

So

$$q_2 = \frac{3}{2}\left(p_2 - \frac{1}{3}p_0\right)$$

Hence

$$q_0(t) = 1,\ q_1(t) = t,\ q_2(t) = \frac{3}{2}\left(t^2 - \frac{1}{3}\right)$$

$$q_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$$

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5. (a) $\begin{vmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 1 = \lambda^2 - 10\lambda + 24$

$\underline{\underline{\lambda_1 = 6}}$, $\underline{\underline{\lambda_2 = 4}}$

$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$D = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix} \quad P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(b) $A = \underset{3\times 3}{U}\ \underset{3\times 2}{\Sigma}\ \underset{2\times 2}{V^T}$ $\boxed{6}$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$\vec{u}_1 = \frac{1}{\sqrt{6}} A\vec{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{3}} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{2} A\vec{v}_2 = \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$A^T \vec{u}_3 = 0$

$$\begin{aligned} 2x_2 + x_3 &= 0 \\ 2x_1 \quad + x_3 &= 0 \end{aligned}$$

$x_1 = -1,\ x_2 = -1,\ x_3 = 2$

$$\vec{u}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}} =$$

$$\begin{bmatrix} \\ \downarrow \\ \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ \sqrt{2} & -\sqrt{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

ⓒ The unit vector $\vec{x}$ for which $A\vec{x}$ is the longest is $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Then

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} A \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \sqrt{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and the length of the last vector is $\sqrt{6}$ as exactly $\sqrt{\lambda_1} = \sqrt{6}$ as expected.