Fall 2016 Math 304 Topics for the exam

On an exam you can be asked to prove any of the theorems stated in the summary.

4.7 Change of bases (in fact: Change of coordinates).

- Know that the matrix M with the property $[\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$**. It is denoted $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ and it is calculated as

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \Big[\, [\mathbf{b}_1]_{\mathcal{C}} \ \cdots \ [\mathbf{b}_n]_{\mathcal{C}} \,\Big];$$

  here $\mathcal{B} = \{\mathbf{b}, \dots, \mathbf{b}_n\}$ and $\mathcal{C}$ are two bases of an n-dimensional vector space $\mathcal{V}$ and $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ denotes the coordinate vector of $\mathbf{v} \in \mathcal{V}$ relative to the basis $\mathcal{B}$.
- $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$
- Know that there is a special basis of $\mathbb{R}^n$, called the **standard basis**, which consists of the columns of the identify matrix I_n. It is denoted by $\mathcal{E}$. This notattion comes from the fact that the vectors in this basis are commonly denoted by $\vec{e}_1, \cdots, \vec{e}_n$.
- Know that for a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ for $\mathbb{R}^n$ (this is a special case) we have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\, \vec{b}_1 \cdots \vec{b}_n \,\big] = P_{\mathcal{B}}$$

  (the matrix $P_{\mathcal{B}}$ was introduced in Section 4.4); this basically says that $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ is a very friendly matrix.
- Know $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\right)\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right) = \left(\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}\right)^{-1}\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \big[\, \vec{c}_1 \cdots \vec{c}_n \,\big]^{-1}\big[\, \vec{b}_1 \cdots \vec{b}_n \,\big]$
  where $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ is another basis for $\mathbb{R}^n$; this basically says that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ can be written as a product of a very friendly matrix and the inverse of another very friendly matrix
- Know Exercises 4 - 10, 13, 14.

5.1 Eigenvectors and eigenvalues.

- Know the definition of an eigenvector and eigenvalue. It is a little tricky. Pay attention.
- Know the definition of an eigenspace and how to find an eigenspace corresponding to a given eigenvalue.
- Know that the eigenvalues of a triangular matrix are the entries on its main diagonal.
- **Theorem.** Eigenvectors corresponding to distinct eigenvalues are linearly independent. (Or, in more formal mathematical language: Let A be an $n{\times}n$ matrix, let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$ and $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$. If $A\vec{v}_k = \lambda_k\vec{v}_k$, $\vec{v}_k \neq \vec{0}$ and $\lambda_j \neq \lambda_k$ for all $j, k = 1, 2, \dots, m$, then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are linearly independent.
- Know the proof of the above theorem for $m = 2$ vectors.

5.2 The characteristic equation.

- Know that λ is an eigenvalue of an $n{\times}n$ matrix A if and only if $\det(A - \lambda I) = 0$
- Know how to calculate $\det(A - \lambda I)$ for $2{\times}2$ matrices, how to find eigenvalues and corresponding eigenvectors. Exercises 1-8, but do more and find eigenvectors as well.

5.3 Diagonalization.

- **Theorem.** (The diagonalization theorem) An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- Know how to decide whether a given 2×2 and 3×3 matrix A, is diagonalizable or not; if it is diagonalizable, how to find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

5.4 Eigenvectors and linear transformations.

- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{C}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the bases $\mathcal{B}$ and $\mathcal{C}$** and it is calculated as
$$M = \Big[\, [T\mathbf{b}_1]_{\mathcal{C}} \ \cdots \ [T\mathbf{b}_n]_{\mathcal{C}} \,\Big],$$
where T be a linear transformation from an m-dimensional vector space $\mathcal{V}$ to an n-dimensional vector space $\mathcal{W}$, $\mathcal{B} = \{\mathbf{b}, \ldots, \mathbf{b}_m\}$ is a basis for $\mathcal{V}$ and $\mathcal{C}$ is a basis for $\mathcal{W}$.
- Know that the matrix M with the property $[T\mathbf{v}]_{\mathcal{B}} = M[\mathbf{v}]_{\mathcal{B}}$ is called the **matrix of T relative to the basis $\mathcal{B}$**, it is denoted by $[T]_{\mathcal{B}}$ and it is calculated as
$$[T]_{\mathcal{B}} = \Big[\, [T\mathbf{b}_1]_{\mathcal{B}} \ \cdots \ [T\mathbf{b}_n]_{\mathcal{B}} \,\Big],$$
where $T : \mathcal{V} \to \mathcal{V}$ is a linear transformation on an n-dimensional vector space $\mathcal{V}$ and $\mathcal{B} = \{\mathbf{b}, \ldots, \mathbf{b}_n\}$ is a basis for $\mathcal{V}$.
- **Theorem.** (The diagonal matrix representation) Let A, D, P be $n\times n$ matrices, where P is invertible and D is diagonal. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ which consists of the columns of P and if $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T\vec{v} = A\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$, then $[T]_{\mathcal{B}} = D$.

5.5 Complex eigenvalues.

- Know how to calculate complex eigenvalues and corresponding eigenvectors of 2×2 and 3×3 real matrices
- Know that the most important class of 2×2 matrices with complex eigenvalues are rotation matrices; the matrix of the counterclockwise rotation about the origin by the angle θ measured in radians relative to the standard basis for $\mathbb{R}^2$ is
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix};$$
the 2×2 matrix R_θ is called the **rotation matrix**.
- **Theorem.** (A "hiding rotation" theorem.) Let A be a real 2×2 matrix with a nonreal eigenvalue. Then there exist an invertible 2×2 matrix P, a positive scalar α and a rotation matrix R_θ such that $A = \alpha P\,R_\theta P^{-1}$.

6.1 Inner product, length, and orthogonality.

- Know the definition of the inner product (dot product) in $\mathbb{R}^n$, its basic properties and calculations involving it.
- Know the definition, the basic properties of the length in $\mathbb{R}^n$ and calculations involving it.
- Know the definition of the distance in $\mathbb{R}^n$ and calculations involving it.
- Know the definition of orthogonality in $\mathbb{R}^n$ and calculations involving it.
- Know the statement and the proof of the linear algebra version of **Pythagorean theorem**.
- Know the definition and the basic properties of the orthogonal complement in $\mathbb{R}^n$.
- Know that for a given $m \times n$ matrix A we have $(\operatorname{Row} A)^{\perp} = \operatorname{Nul} A$ and $(\operatorname{Col} A)^{\perp} = \operatorname{Nul} A^{\top}$.
- Know the geometric interpretation of the inner product (dot product) in $\mathbb{R}^2$ and $\mathbb{R}^3$:
$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\vartheta, \tag{1}$$
where $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^2$ or in $\mathbb{R}^3$, ϑ is the angle at the vertex O in the triangle OAB with O being the origin, A being the endpoint of $\vec{u}$ and B the endpoint of $\vec{v}$. (You should know the proof of formula (1).)

6.2 Orthogonal sets.

- Know the definition of an orthogonal set of vectors.
- **Theorem.** (Linear independence of orthogonal sets.) Let $\mathcal{S} = \{\vec{u}_1, \dots, \vec{u}_m\}$ be a subset of $\mathbb{R}^n$. If $\mathcal{S}$ is an orthogonal set which consists of nonzero vectors, then $\mathcal{S}$ is linearly independent.
- Know the definition of an orthogonal bases.
- **Theorem.** (Easy expansions with orthogonal bases.) Let $\{\vec{u}_1, \dots, \vec{u}_m\}$ be an orthogonal basis of a subspace $\mathcal{W}$ of $\mathbb{R}^n$. Then for every $\vec{y} \in \mathcal{W}$ we have

$$\vec{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\,\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\,\vec{u}_2 + \cdots + \frac{\vec{y}\cdot\vec{u}_m}{\vec{u}_m\cdot\vec{u}_m}\,\vec{u}_m$$

- Know the definition of the orthogonal projection of a vector $\vec{y}$ onto a vector $\vec{u}$: A vector $\widehat{\vec{y}} = \alpha\vec{u}$ is called the **orthogonal projection of $\vec{y}$ onto $\vec{u}$** if the difference $\vec{y} - \widehat{\vec{y}}$ is orthogonal to $\vec{u}$. (Convince yourself that

$$\widehat{\vec{y}} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\,\vec{u}$$

is the orthogonal projection of $\vec{y}$ onto $\vec{u}$.)
- Know how to do calculations with orthogonal projections.
- The definitions of orthonormal set of vectors, orthonormal basis, a matrix with orthonormal columns.
- Know the characterization of a matrix with orthonormal columns: The columns of U are orthonormal if and only if $U^\top U = I$. (Please make sure that you understand the order of the matrix and its transpose in the previous identity.)
- Know the properties of matrices with orthonormal columns.

6.3 Orthogonal projections.

- Know the definition of the orthogonal projection of a vector $\vec{y}$ onto a subspace $\mathcal{W}$: A vector $\widehat{\vec{y}} \in \mathcal{W}$ is called the **orthogonal projection of $\vec{y}$ onto $\mathcal{W}$** if the difference $\vec{y} - \widehat{\vec{y}}$ is orthogonal to $\mathcal{W}$.
- **Theorem.** (The orthogonal decomposition theorem.) Let $\mathcal{W}$ be a subspace of $\mathbb{R}^n$. Then each $\vec{y}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\vec{y} = \widehat{\vec{y}} + \vec{z}$$

where $\widehat{\vec{y}} \in \mathcal{W}$ and $\vec{z} \in \mathcal{W}^\perp$. If $\{\vec{u}_1, \dots, \vec{u}_m\}$ is an orthogonal basis for $\mathcal{W}$, then

$$\widehat{\vec{y}} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\,\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\,\vec{u}_2 + \cdots + \frac{\vec{y}\cdot\vec{u}_m}{\vec{u}_m\cdot\vec{u}_m}\,\vec{u}_m \tag{2}$$

- Know that equation (2) simplifies if we assume that $\{\vec{u}_1, \dots, \vec{u}_m\}$ is an **orthognormal basis** for $\mathcal{W}$; then

$$\widehat{\vec{y}} = (\vec{y}\cdot\vec{u}_1)\,\vec{u}_1 + (\vec{y}\cdot\vec{u}_2)\,\vec{u}_2 + \cdots + (\vec{y}\cdot\vec{u}_m)\,\vec{u}_m. \tag{3}$$

- Know the amazing fact that equation (3) can be written as a matrix equation; let

$$U = \begin{bmatrix}\vec{u}_1 & \cdots & \vec{u}_m\end{bmatrix}$$

be a matrix with **orthonormal columns**, then

$$\widehat{\vec{y}} = (\vec{y}\cdot\vec{u}_1)\,\vec{u}_1 + (\vec{y}\cdot\vec{u}_2)\,\vec{u}_2 + \cdots + (\vec{y}\cdot\vec{u}_m)\,\vec{u}_m = U\,U^\top\vec{y}.$$

The explanation for the last equality based on the definition of the matrix multiplication is as follows:

$$(\vec{y}\cdot\vec{u}_1)\,\vec{u}_1 + \cdots + (\vec{y}\cdot\vec{u}_m)\,\vec{u}_m = \begin{bmatrix}\vec{u}_1 & \cdots & \vec{u}_m\end{bmatrix}\begin{bmatrix}\vec{y}\cdot\vec{u}_1\\ \vdots \\ \vec{y}\cdot\vec{u}_m\end{bmatrix} = U\begin{bmatrix}(\vec{u}_1)^\top\vec{y}\\ \vdots \\ (\vec{u}_m)^\top\vec{y}\end{bmatrix} = U\begin{bmatrix}(\vec{u}_1)^\top\\ \vdots \\ (\vec{u}_m)^\top\end{bmatrix}\vec{y} = U\,U^\top\vec{y}.$$

6.4 The Gram-Schmidt orthogonalization.

➢ Know the Gram-Schmidt orthogonalization process: Let m and n be positive integers such that $2 \leq m \leq n$. Let $\bigl\{\vec{x}_1, \dots, \vec{x}_m\bigr\}$ be a basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$. The vectors $\vec{v}_1, \dots, \vec{v}_m$ recursively defined by

$$
\begin{aligned}
\vec{v}_1 &= \vec{x}_1, \\
\vec{v}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\,\vec{v}_1, \\
\vec{v}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\,\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\,\vec{v}_2, \\
&\;\vdots \\
\vec{v}_m &= \vec{x}_m - \frac{\vec{x}_m \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\,\vec{v}_1 - \frac{\vec{x}_m \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\,\vec{v}_2 - \cdots - \frac{\vec{x}_m \cdot \vec{v}_{m-1}}{\vec{v}_{m-1} \cdot \vec{v}_{m-1}}\,\vec{v}_{m-1},
\end{aligned}
$$

have the following properties

(i) $\bigl\{\vec{v}_1, \dots, \vec{v}_m\bigr\}$ is an orthogonal basis for $\mathcal{W}$.

(ii) For all $k \in \{1, \dots, m\}$ we have $\operatorname{Span}\bigl\{\vec{v}_1, \dots, \vec{v}_k\bigr\} = \operatorname{Span}\bigl\{\vec{x}_1, \dots, \vec{x}_k\bigr\}$.